Cauchy Sequences and Complete Metric Spaces

Let's first consider two examples of convergent sequences in $\mathbb{R}$:

Example 1: Let $x_n = \frac{1}{n}\sqrt{2}$ for each $n \in \mathbb{N}$. Note that each x_n is an irrational number (*i.e.*, $x_n \in \mathbb{Q}^c$) and that $\{x_n\}$ converges to 0. Thus, $\{x_n\}$ converges in $\mathbb{R}$ (*i.e.*, to an element of $\mathbb{R}$). But 0 is a rational number (thus, $0 \notin \mathbb{Q}^c$), so although the sequence $\{x_n\}$ is entirely in $\mathbb{Q}^c$, it does not converge in $\mathbb{Q}^c$, in spite of being well-behaved in the sense that it converges in $\mathbb{R}$.

Example 2: Let $x_1 \in \mathbb{N}$, and let x_n be the sequence defined by $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ for each $n \in \mathbb{N}$. We can show that the sequence $\{x_n\}$ converges to $\sqrt{2}$. (You'll be asked to do that in an exercise below.) Then $\{x_n\}$ is a sequence of rational numbers that converges to the irrational number $\sqrt{2}$ — *i.e.*, each x_n is in $\mathbb{Q}$ and $\lim\{x_n\} = \sqrt{2} \notin \mathbb{Q}$. Thus, in a parallel to Example 1, $\{x_n\}$ here converges in $\mathbb{R}$ but does not converge in $\mathbb{Q}$.

Examples 1 and 2 demonstrate that both the set $\mathbb{Q}^c$ of irrational numbers and the set $\mathbb{Q}$ of rational numbers are not entirely well-behaved metric spaces: there are well-behaved sequences in each space that don't converge to an element of the space. The sequences are well-behaved in the sense that they do converge in $\mathbb{R}$. The following definition formalizes this idea of a well-behaved sequence in a metric space (such as $\mathbb{Q}$ and $\mathbb{Q}^c$), but without requiring any reference to some other, larger metric space (such as $\mathbb{R}$).

Definition: A sequence $\{x_n\}$ in a metric space (X, d) is **Cauchy** if

$$\forall \epsilon > 0 : \exists \overline{n} \in \mathbb{N} : \ m, n > \overline{n} \Rightarrow d(x_m, x_n) < \epsilon.$$

Remark: Convergent sequences are Cauchy.

Proof:

Let $\{x_n\} \to \overline{x}$, let $\epsilon > 0$, let $\overline{n}$ be such that $n > \overline{n} \Rightarrow d(x_n, \overline{x}) < \epsilon/2$, and let $m, n > \overline{n}$. Then

$$d(x_m, \overline{x}) < \frac{\epsilon}{2} \ \text{ and } \ d(x_n, \overline{x}) < \frac{\epsilon}{2},$$

and the Triangle Inequality yields

$$d(x_m, x_n) \leqq d(x_m, \overline{x}) + d(x_n, \overline{x}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \blacksquare$$

Exercise: The real sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ converges, so it's Cauchy. Prove directly that it's Cauchy, by showing how the $\overline{n}$ in the definition depends upon ϵ.

Definition: A metric space (X, d) is **complete** if every Cauchy sequence in X converges in X (*i.e.*, to a limit that's in X).

Example 3: The real interval $(0, 1)$ with the usual metric is not a complete space: the sequence $x_n = \frac{1}{n}$ is Cauchy but does not converge to an element of $(0, 1)$.

Example 4: The space $\mathbb{R}^n$ with the usual (Euclidean) metric is complete. We haven't shown this yet, but we'll do so momentarily.

Remark 1: Every Cauchy sequence in a metric space is bounded.

Proof: Exercise.

Remark 2: If a Cauchy sequence has a subsequence that converges to $\overline{x}$, then the sequence converges to $\overline{x}$.

Proof: Exercise.

In order to prove that $\mathbb{R}$ is a complete metric space, we'll make use of the following result:

Proposition: Every sequence of real numbers has a monotone subsequence.

Proof: Suppose the sequence $\{x_n\}$ has no monotone increasing subsequence; we show that then it must have a monotone decreasing subsequence. The sequence $\{x_n\}$ must have a first term, say x_{n_1}, such that all subsequent terms are at least as small (*i.e.*, $n > n_1 \Rightarrow x_n \leqq x_{n_1}$); otherwise $\{x_n\}$ would have a monotone increasing subsequence. Similarly, the subsequence $\{x_{n_1+1}, x_{n_1+2}, \ldots\}$ must have a first term x_{n_2} such that all subsequent terms are at least as small; note that $x_{n_1} \geqq x_{n_2}$. Continuing for $n_1, n_2, n_3, \ldots$, we have a subsequence $\{x_{n_k}\}$ such that $x_{n_1} \geqq x_{n_2} \geqq x_{n_3} \geqq \ldots$, a monotone decreasing subsequence. ■

Now we'll prove that $\mathbb{R}$ is a complete metric space, and then use that fact to prove that the Euclidean space $\mathbb{R}^n$ is complete.

Theorem: $\mathbb{R}$ is a complete metric space — *i.e.*, every Cauchy sequence of real numbers converges.

Proof: Let $\{x_n\}$ be a Cauchy sequence. Remark 1 ensures that the sequence is bounded, and therefore that every subsequence is bounded. The proposition we just proved ensures that the sequence has a monotone subsequence. The Monotone Convergence Theorem ensures that this bounded monotone subsequence converges. And therefore Remark 2 ensures that the original sequence converges. ■

This proof used the Completeness Axiom of the real numbers — that $\mathbb{R}$ has the LUB Property — via the Monotone Convergence Theorem. We could have gone instead in the other direction: taking "every Cauchy sequence of real numbers converges" to be the Completeness Axiom, and then proving that $\mathbb{R}$ has the LUB Property.

Theorem: The normed vector space $\mathbb{R}^n$ is a complete metric space.

Proof: Exercise.

Example 5: The closed unit interval $[0,1]$ is a complete metric space (under the absolute-value metric). This is easy to prove, using the fact that $\mathbb{R}$ is complete.

Example 6: The space $C[0,1]$ is complete. (We haven't shown this yet.)

Exercise: In a previous exercise set we worked with a sequence of distribution functions F_n defined by

$$F_n(x) = \begin{cases} nx, & \text{if } x \leqq \frac{1}{n} \\ 1\,, & \text{if } x \geqq \frac{1}{n}. \end{cases}$$

on the unit interval $[0,1]$ in $\mathbb{R}$. We showed that $\{F_n\}$ does not converge in $C[0,1]$. Therefore, if $\{F_n\}$ were Cauchy, $C[0,1]$ would not be complete. Verify that $\{F_n\}$ is *not* Cauchy.

Example 7: (Obtaining $\mathbb{R}$ as the completion of $\mathbb{Q}$.)
Let S be the set of Cauchy sequences in $\mathbb{Q}$ — *i.e.*, the set of Cauchy sequences of rational numbers — with the usual metric. Define a relation $\sim$ on S as follows:

$$\{x_n\} \sim \{x'_n\} \;\text{ if }\; \forall\epsilon > 0 : \exists\bar{n} \in \mathbb{N} : m, n > \bar{n} \Rightarrow d(x_m, x'_n) < \epsilon$$

Let $\mathbb{Q}^* = S/\sim$, the partition of S consisting of equivalence classes of Cauchy sequences. Define the distance function d^* for $\mathbb{Q}^*$ as follows:

For any $x, x' \in \mathbb{Q}^*$, let $\{x_n\} \in x$ and $\{x'_n\} \in x'$ (*i.e.*, $x = [\{x_n\}]$ and $x' = [\{x'_n\}]$).

Then define $d^*(x, x')$ by $d^*(x, x') = \lim_{n\to\infty} d(x_n, x'_n)$.

It's pretty straightforward to show that d^* is well-defined and is a metric for $\mathbb{Q}^*$. The metric space $(\mathbb{Q}^*, d^*)$ can be placed into one-to-one correspondence with $(\mathbb{R}, |\cdot|)$, each constant sequence $\{r, r, r, \ldots\}$ of rationals corresponding to the rational number $r \in \mathbb{Q} \subseteq \mathbb{R}$. The set $\mathbb{Q}^*$ is one way of defining $\mathbb{R}$.

Exercise: Verify that the relation $\sim$ defined in Example 7 is an equivalence relation.

Example 8: (The Completion of a Metric Space)
Let (X, d) be a metric space that is not complete. Just as in Example 7, let S be the set of Cauchy sequences in X; define the equivalence relation $\sim$ in the same way, and let X^* be the quotient space $S/\sim$; and define d^* on the quotient space X^* in the same way as in Example 7. Then we can show, just as in Example 7, that d^* is well-defined and is a metric for X^*; that (X^*, d^*) is a complete metric space; and that X corresponds to a subset of X^* — we say that X **is embedded in** X^*. The complete metric space (X^*, d^*) is called the **completion** of (X, d).

Example 9: The open unit interval $(0,1)$ in $\mathbb{R}$, with the usual metric, is an incomplete metric space. What is its completion, $((0,1)^*, d^*))$?

Theorem: A subset of a complete metric space is itself a complete metric space if and only if it is closed.

Proof: Exercise.

Recall that every normed vector space is a metric space, with the metric $d(x,x') = \|x - x'\|$. Therefore our definition of a complete metric space applies to normed vector spaces: an n.v.s. is complete if it's complete as a metric space, *i.e.*, if all Cauchy sequences converge to elements of the n.v.s.

Definition: A complete normed vector space is called a **Banach space**.

Example 4 revisited: $\mathbb{R}^n$ with the Euclidean norm is a Banach space.

Example 5 revisited: The unit interval $[0,1]$ is a complete metric space, but it's not a Banach space because it's not a vector space.

Example 6 revisited: $C[0,1]$ is a Banach space.

Exercise: Let a and x_1 be positive real numbers, and let $\{x_n\}$ be the sequence defined by $x_{n+1} = \frac{1}{2}x_n + \frac{a}{2x_n}$ for each $n \in \mathbb{N}$. Verify that $\{x_n\}$ converges to $\sqrt{a}$. **Hint:** You'll probably find it helpful to remember that the convergent sequences comprise a vector subspace of the vector space $\mathbb{R}^\infty$ of all real sequences, and to remember the algebra of limits of sequences: $\lim\{x_n + y_n\} = \lim\{x_n\} + \lim\{y_n\}$, etc. But in order for you to use these algebraic properties you also need to know that the sequences in question actually do converge. For that you might want to use the Monotone Convergence Theorem.

Sequences defined recursively, like the sequence in the above exercise, are important in economics. We'll see sequences like this later in this course when we study fixed point theorems and their application to the Nash equilibria of games and to growth theory. They'll appear in Economics 501B when we study computation of market equilibria and convergence to equilibrium.