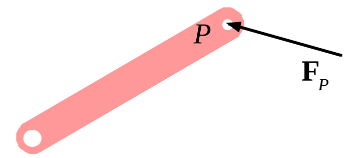
7. FORCE ANALYSIS

This chapter discusses some of the methodologies used to perform force analysis on mechanisms. The chapter begins with a review of some fundamentals of force analysis using vectors. Then a review of graphical and analytical methods of force analysis on stationary mechanisms, known as static force analysis, is provided. Finally, force analysis of mechanisms in motion, known as dynamic force analysis, will be discussed.

Fundamentals

Force Vector

A force that acts on a point of a link carries the index of the point. For example $\mathbf{F}_P$.



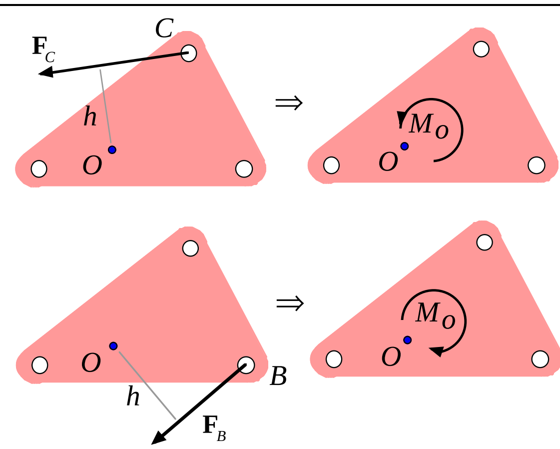
Moment About A Point

In planar systems, the moment of a force about an arbitrary point is a moment vector along an axis perpendicular to the plane (*z*-axis). For example, the moment of the force $\mathbf{F}_C$ about *O* is a moment in the positive *z*-direction (CCW) with a magnitude

$$M_O = hF_C$$

where *h* is the distance from *O* to the axis of the force, also called the *moment arm*. In the second example, the moment of $\mathbf{F}_B$ about *O* is a moment in the negative *z*-direction (CW) with a magnitude

$$M_O = hF_B$$

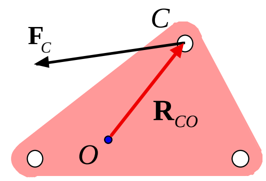


The direction of a moment can be determined using our right-hand—the thumb would indicate the direction of the moment when the other four fingers are curled about the point in the direction of the force.

The moment of a force about a point can also be determined using the vector-product operation. For example, the moment of the force $\mathbf{F}_C$ about *O* is determined as

$$\mathbf{M}_O = \mathbf{R}_{CO} \times \mathbf{F}_C$$

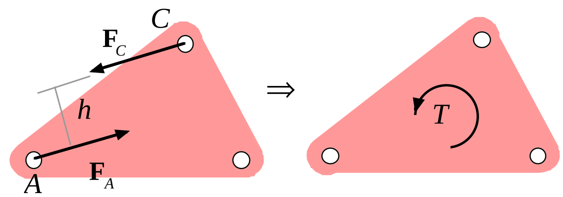
A short review of the vector-product operation is provided at the end of this introductory section.



Force Couples and Torques

Two parallel forces, equal in magnitude and opposite in direction, acting on two different points of a link form a *couple*. The moment of a couple, called a *torque*, is a vector in the *z*-direction and its magnitude is

$$T = hF$$



where *h* is the distance between the two axes and $F = F_A = F_C$ is the magnitude of either force. The positive or negative direction of the torque can be determined based on the right-hand method.

Common Forces and Torques

Forces and torques (moments) that act on a link can be the result of gravity, springs, dampers, actuators, friction, etc. These forces and torques can also be the result of reaction forces or reaction torques from other links. These forces and torques can be categorized as applied, reaction, and friction.

Applied forces and torques

These are either known constants (gravity for example), or functions of positions (springs), or functions of positions and velocities (dampers).

Reaction forces and torques

These are functions of applied forces/torques in static problems, and functions of the applied forces/torques and accelerations in dynamics problems.

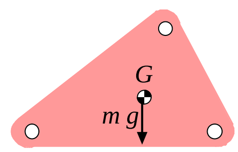
Friction forces and torques

These may appear in machines as viscous (wet) or Coulomb (dry). *Viscous friction* depends on velocities; therefore it can be categorized as an applied force/torque. *Coulomb friction* depends on reaction forces and possibly velocities; therefore it can be categorized as a reaction force/torque.

Applied Forces and Torques

Gravity

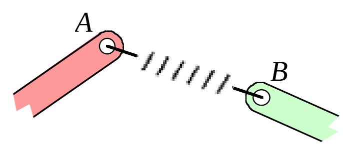
The weight of a body is applied as a force in the direction of gravity at the mass center.



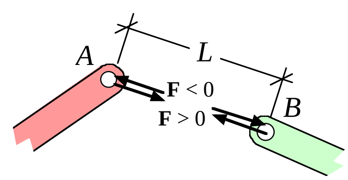
Point-to-point spring

The formula to determine the force of a linear spring is

$$F = k(L - L^0)$$



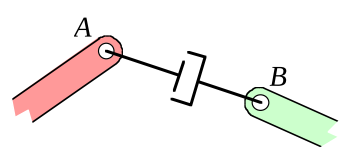
where k is the stiffness, L is the deformed length, and L^0 is the undeformed length of the spring. The deformed length, L, must be computed based on the instantaneous positions (coordinates) of the two attachment points. If the computed force is negative, the spring is in compression. If the force is positive, the spring is in tension—the pair of forces must be applied to the two links accordingly as shown.



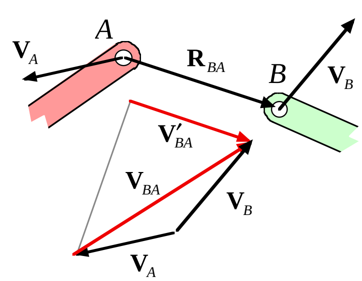
Point-to-point damper

The formula to compute the force of a linear damper is

$$F = c\,\dot{L}$$

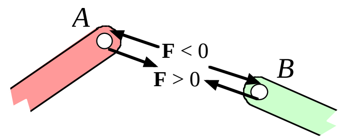


where c is the damping coefficient, and $\dot{L}$ is the time rate of change in the damper's length. $\dot{L}$ must be computed based on the velocities of the attachment points. As shown in the diagram, the relative velocity $\mathbf{V}_{BA} = \mathbf{V}_B - \mathbf{V}_A$ is first determined and then projected along the axis of the damper to obtain $\mathbf{V}'_{BA}$. The magnitude of this vector is the magnitude of $\dot{L}$. If $\mathbf{V}'_{BA}$ and $\mathbf{R}_{BA}$ are in the same direction, $\dot{L}$ is positive (the



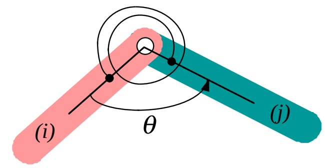
damper is increasing its length), otherwise $\dot{L}$ must be given a negative sign (the damper is shortening).

The computed damper force is applied as a pair of forces to the attachment points, in the opposite directions, depending on the sign of F as shown.



Rotational spring (or damper)

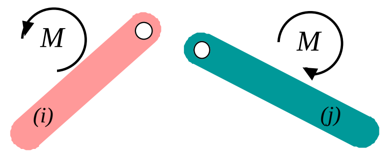
A rotational (also called torsional) spring is attached between two links about the axis of a pin joint. Two axes originating from the center of the pin joint, one on each link, are defined and the angle between them is measured.



The torque for a rotational spring is computed as

$$T = k(\theta - \theta^0)$$

where k is the stiffness, θ is the deformed angle, and θ^0 is the undeformed angle of the spring.



A pair of torques, one on each link, is applied in opposite directions.

The formula to compute the torque of a rotational damper is

$$T = c\dot{\theta}$$

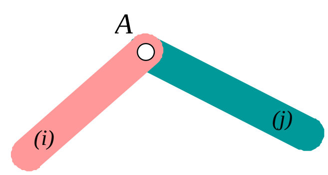
where c is the damping coefficient, and $\dot{\theta}$ is the time rate of change in the damper's angle. $\dot{\theta}$ can be computed based on the angular velocities of the two bodies as $\dot{\theta} = \omega_j - \omega_i$. Whether $\dot{\theta}$ is positive (increasing angle) or negative (decreasing angle), the pair of torques that are applied to the two bodies must oppose the motion.

Reaction Forces Torques

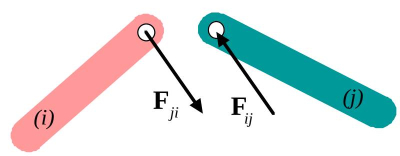
Two links connected by a kinematic joint apply reaction forces (and/or torques) on one another.

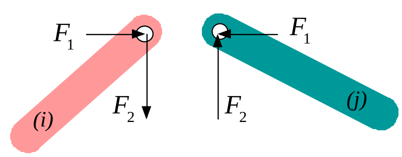
Pin joint

Two links connected by a pin joint apply reaction forces on each other. The reaction forces are equal in magnitude and opposite in direction. The magnitude and directions that are shown on the free-body-diagrams are arbitrary—they must be determined through an analysis.



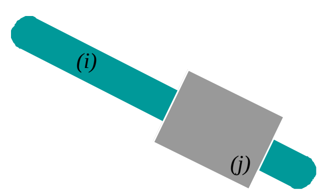
The reaction force on each link can be represented in term of its x and y components, such as $F_{ji(x)}$ and $F_{ji(y)}$. For notational simplicity, we will use a single index to show each component; for example, F_1 and F_2. If the assigned direction to a component is determined to be correct through an analysis, the solution for that component will come out with a positive sign. Otherwise the solution will end up with a negative sign indicating that the assumed direction for the component must be reversed.





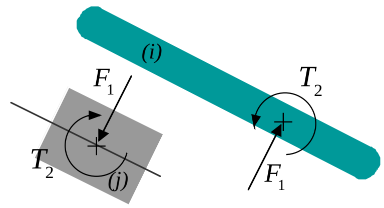
Sliding joint

Two links connected by a sliding joint apply reaction forces and torques on each other. The reaction forces, $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, are equal in magnitude, opposite in direction, and act perpendicular to the axis of the joint. Although these are distributed forces that act along the surfaces of contact,



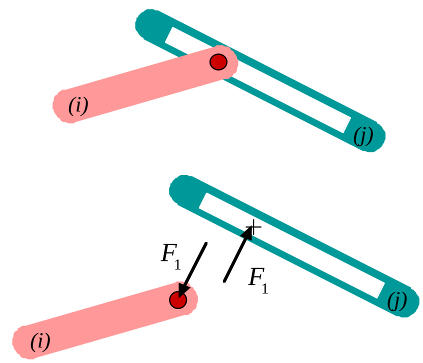
in our analyses we place them as concentrated forces at the center of the block. The reaction torques, T_{ij} and T_{ji}, are equal in magnitude and opposite in direction.

In our free-body diagrams, each reaction force or torque is denoted with a single index for convenience. For example, F_1 and T_2. The correct direction for these components will be determined through an analysis.



Pin-sliding joint

Two links connected by a pin-sliding joint apply reaction forces on each other. On the free-body diagrams of the links, the reaction forces are shown equal in magnitude, opposite in direction, acting perpendicular to the axis of the joint.



Scalar and Vector Products

In static and dynamic analysis, we often encounter vector operations such as scalar product or vector product. In this section a short review on how to evaluate such products is presented.

Scalar (dot) product

The scalar product of two vectors, such as **F** and **V**, can be determined either analytically or graphically.

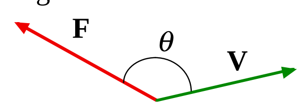
Analytical:

The analytical scalar product can be computed in two ways depending on how the vectors are defined:

(a) The magnitudes of the vectors are known as F and V, and the angle between the two vectors is known as θ. The scalar product is computed as

$$\mathbf{F} \bullet \mathbf{V} = FV\cos\theta$$

The angle can be measured from **F** to **V** or vice versa.

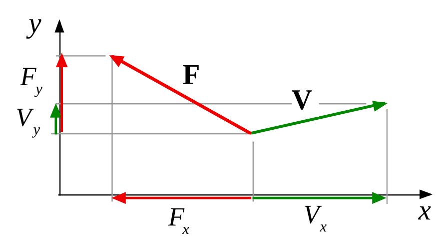


(b) The vectors are known in component form:

$$\mathbf{F} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}, \quad \mathbf{V} = \begin{Bmatrix} V_x \\ V_y \end{Bmatrix}$$

The scalar product is computed as

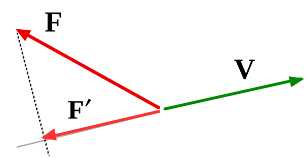
$$\mathbf{F} \bullet \mathbf{V} = F_x V_x + F_y V_y$$



Graphical:

In order to determine the scalar product of two vectors graphically, one of the vectors must be projected onto the axis of the other vector. For example, **F** is projected onto the axis of **V** to obtain $\mathbf{F}'$. The magnitudes of $\mathbf{F}'$ and **V** are measured and denoted as F' and V. Then the scalar product is computed as: $\mathbf{F} \bullet \mathbf{V} = F'V$

If F' and V are in the same direction, the product is considered positive, otherwise the product must be considered negative.



Vector (cross) product

The vector product of two vectors can be determined either analytically or graphically.

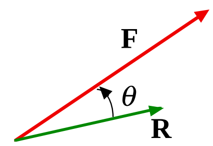
Analytical:

The vector product can be computed in two ways depending on how the vectors are defined:

(a) The magnitudes vectors **R** and **F** are R and F, and he angle between the two vectors is θ.

The magnitude of the vector product is computed as: $\mathbf{R} \times \mathbf{F} = RF \sin\theta$

The angle must be measured from **R** to **F**, CCW.



(b) The vectors are known in component form:

$$\mathbf{R} = \begin{Bmatrix} R_x \\ R_y \end{Bmatrix}, \quad \mathbf{F} = \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$$

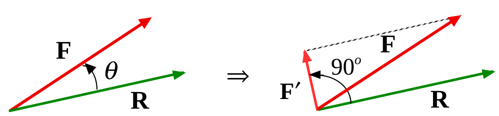
The vector product is computed as: $\mathbf{R} \times \mathbf{F} = \breve{\mathbf{R}} \bullet \mathbf{F} = R_x F_y - R_y F_x$

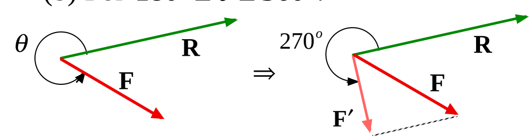
Graphical:

In order to determine the vector product of vectors **R** and **F** graphically, vector **F** is projected onto an axis perpendicular to **R** to obtain $\mathbf{F}'$. We measure the magnitudes of **R** and $\mathbf{F}'$; i.e., R and F':

(a) For $0 \le \theta \le 180^\circ$: $\mathbf{R} \times \mathbf{F} = R \cdot F'$

(b) For $180^\circ \le \theta \le 360^\circ$: $\mathbf{R} \times \mathbf{F} = -R \cdot F'$





Whether the vector product operation is carried out analytically or graphically, the resultant vector is perpendicular to the plane of the original two vectors. In planar problems, the resultant vector will be along the *z*-axis. Its direction depends on the angle between the two original vectors, measured from the first vector CCW towards the second vector.

Example

Two vectors are given as **A** and **B** with the magnitudes A = 2.0 and B = 3.0, and angles $\theta_A = 30^o$ and $\theta_B = 135^o$. In component form these vectors can be described as

$$\mathbf{A} = \begin{Bmatrix} 2.0\cos 30 \\ 2.0\sin 30 \end{Bmatrix} = \begin{Bmatrix} 1.73 \\ 1.0 \end{Bmatrix}, \quad \mathbf{B} = \begin{Bmatrix} 3.0\cos 135 \\ 3.0\sin 135 \end{Bmatrix} = \begin{Bmatrix} -2.12 \\ 2.12 \end{Bmatrix}$$

Scalar product

The angle between these two vectors is $\theta_{AB} = 105^o$. The scalar product can be computed as

$$\mathbf{A} \bullet \mathbf{B} = (2.0)(3.0)\cos 105 = -1.55 \qquad \text{(a)}$$

Using the component form of the vectors we have

$$\mathbf{A} \bullet \mathbf{B} = (1.73)(-2.12) + (1.0)(2.12) = -1.55 \qquad \text{(b)}$$

We can project **B** onto the axis of **A** and measure the magnitude of the projected vector to be $B' = 0.77$. Since this projected vector is in the opposite direction of **A**, we have

$$\mathbf{A} \bullet \mathbf{B} = -(0.77)(2) = -1.55 \qquad \text{(c)}$$

Vector product

We compute the magnitude of the cross product of the two vectors as

$$\mathbf{A} \times \mathbf{B} = (2.0)(3.0)\sin 105 = 5.79 \qquad \text{(d)}$$

Since the answer is positive, the resultant vector is in the positive *z* direction (coming out of the plane). We can also use the components to get

$$\mathbf{A} \times \mathbf{B} = (1.73)(-2.12) + (1.0)(-2.12) = 5.79 \qquad \text{(e)}$$

Projecting **B** onto an axis perpendicular to **A** and measuring the magnitude of the projected vector provides $B' = 2.89$, which yields

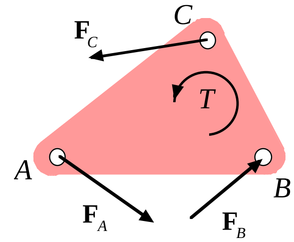
$$\mathbf{A} \times \mathbf{B} = (2.89)(2) = 5.79$$

Static Force Analysis

Static Equilibrium Equations

A body is said to be in static equilibrium if under a set of applied forces and torques its translational (linear) and rotational accelerations are zeros (a body could be stationary or in motion with a constant linear velocity).

Planar static equilibrium equations for a single body that is acted upon by forces and torques are expressed as



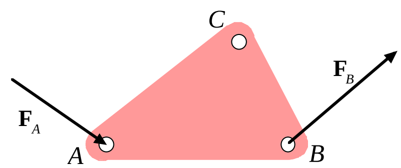
$$\sum \mathbf{F}_i = \mathbf{0} \quad \Rightarrow \quad \begin{cases} \sum F_{i(x)} = 0 \\ \sum F_{i(y)} = 0 \end{cases} \qquad \text{(se.1)} \quad \text{(Force equation)}$$

$$\sum T_j + \sum M_i = 0 \qquad \text{(se.2)} \quad \text{(Moment equation)}$$

Equation (se.1) represents the sum of all the forces acting on the link, and Eq. (se.2) represents the sum of all the torques, T_j, and the moments, M_i, caused by all the forces acting on the body with respect to any reference point.

Example

Assume three forces and a torque act on a body. Two of the forces, $\mathbf{F}_A$ and $\mathbf{F}_B$, are known as shown. Determine $\mathbf{F}_C$ and the applied torques T in order for the body to stay in static equilibrium. Consider $F_A = 1.5$, $\theta_{F_A} = 325^o$, $F_B = 1.9$, and $\theta_{F_B} = 40^o$.

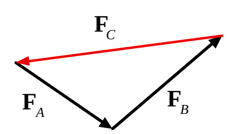


Sum of forces

Equation (se.1) can be solved either graphically or analytically.

Graphical

Equation (se.1) states that the sum of forces that act on the body must form a closed loop; i.e., $\mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = \mathbf{0}$. We construct this vector sum graphically in order to determine $\mathbf{F}_C$. Direct measurement from the figure yields $F_C = 2.7$ and $\theta_{F_C} = 187^o$.



Analytical

Equation (se.1) is projected onto the *x*- and *y*-axes to obtain two algebraic equations:

$$\sum_{i=1}^{n} F_i \cos\theta_{F_i} = 0 \quad \Rightarrow \quad F_A \cos\theta_{F_A} + F_B \cos\theta_{F_B} + F_C \cos\theta_{F_C} = 0$$

$$\sum_{i=1}^{n} F_i \sin\theta_{F_i} = 0 \quad \Rightarrow \quad F_A \sin\theta_{F_A} + F_B \sin\theta_{F_B} + F_C \sin\theta_{F_C} = 0$$

To solve for $\mathbf{F}_C$, we write the equations as

$$F_C \cos\theta_{F_C} = -F_A \cos\theta_{F_A} - F_B \cos\theta_{F_B}$$

$$F_C \sin\theta_{F_C} = -F_A \sin\theta_{F_A} - F_B \sin\theta_{F_B}$$

We square both sides and add the equations to get

$$F_C = \left((-F_A \cos\theta_{F_A} - F_B \cos\theta_{F_B})^2 + (-F_A \sin\theta_{F_A} - F_B \sin\theta_{F_B})^2 \right)^{1/2}$$

Then sine and cosine of θ_{F_C} can be computed as

$$\cos\theta_{F_C} = -(F_A \cos\theta_{F_A} + F_B \cos\theta_{F_B}) / F_C \;, \quad \sin\theta_{F_C} = -(F_A \sin\theta_{F_A} + F_B \sin\theta_{F_B}) / F_C$$

These expressions provide the value of θ_C in its correct quadrant.

Numerically, the two known vectors are expressed in component form as

$$\mathbf{F}_A = \begin{Bmatrix} 1.23 \\ -0.86 \end{Bmatrix}, \; \mathbf{F}_B = \begin{Bmatrix} 1.46 \\ 1.22 \end{Bmatrix}$$

Then vector $\mathbf{F}_C$ can be computed as

$$\mathbf{F}_C = -\mathbf{F}_A - \mathbf{F}_B = -\begin{Bmatrix} 1.23 \\ -0.86 \end{Bmatrix} - \begin{Bmatrix} 1.46 \\ 1.22 \end{Bmatrix} = \begin{Bmatrix} -2.68 \\ -0.36 \end{Bmatrix}$$

The magnitude of this vector is $F_C = \sqrt{(-2.68)^2 + (-0.36)^2} = 2.7$. The vector can also be expressed as

$$\mathbf{F}_C = 2.7 \begin{Bmatrix} -2.68 / 2.7 \\ -0.36 / 2.7 \end{Bmatrix} = 2.7 \begin{Bmatrix} -0.99 \\ -0.13 \end{Bmatrix}$$

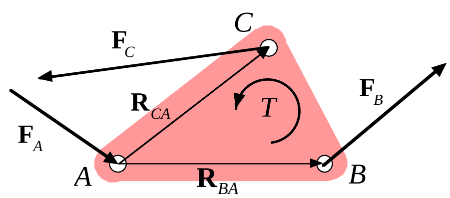
For the angle of this vector we have $\cos\theta_{F_C} = -0.99$ and $\sin\theta_{F_C} = -0.13$. These values yield $\theta_{F_C} = 187^o$.

Note: If you use Matlab to compute the angle, take advantage of the function `atan2`. In this problem the function can be used as `theta_F_C = atan2(-0.13, -0.99)`. The function provides the angle in its correct quadrant.

Sum of moments

For equation (se.2), the sum of the torque and the moments of all three forces about any arbitrary point must be equal to zero. We take the sum of moments about *A*; i.e.,

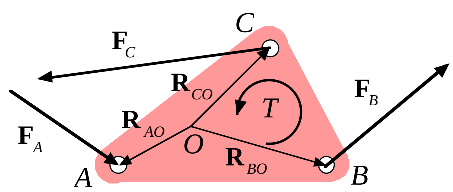
$$T + \mathbf{R}_{BA} \times \mathbf{F}_B + \mathbf{R}_{CA} \times \mathbf{F}_C = 0$$



Note that the moment arm for $\mathbf{F}_A$ is a zero vector. This equation is re-written as

$$T = -\mathbf{R}_{BA} \times \mathbf{F}_B - \mathbf{R}_{CA} \times \mathbf{F}_C = -\breve{\mathbf{R}}_{BA} \bullet \mathbf{F}_B - \breve{\mathbf{R}}_{CA} \bullet \mathbf{F}_C$$

We can also write the moment equation with respect to point *B*, *C* or any other point in the same way. For example, if we pick an arbitrary point, such as *O*, the moment equation becomes:



$$T + \mathbf{R}_{AO} \times \mathbf{F}_A + \mathbf{R}_{BO} \times \mathbf{F}_B + \mathbf{R}_{CO} \times \mathbf{F}_C = 0$$

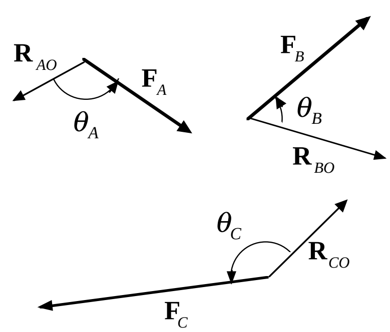
This equation yields the unknown torque as:

$$T = -\mathbf{R}_{AO} \times \mathbf{F}_A - \mathbf{R}_{BO} \times \mathbf{F}_B - \mathbf{R}_{CO} \times \mathbf{F}_C$$

Any of the above moment equations can be solved either graphically or analytically. In this example we take the moment equation about *O*.

Graphical:

We first need to determine the angle between every **R** vector and its corresponding **F** vector. For the purpose of clarity, each pair of **R** and **F** vectors is shown separately with the corresponding angle. The angles are measured (from ***R*** to ***F*** CCW) and the moment equation is evaluated as



$$T + R_{AO} F_A \sin\theta_A + R_{BO} F_B \sin\theta_B + R_{CO} F_C \sin\theta_C = 0$$

From the figure we measure the following magnitudes: $R_{AO} = 1.1$, $R_{BO} = 1.8$ and

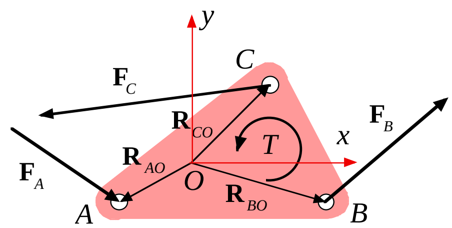
$R_{CO} = 1.5$. The angles between each force vector and its corresponding position vector are measured to be: $\theta_A = 128^o$, $\theta_B = 58^o$ and $\theta_C = 143^o$. With these values the applied torque is determined to be $T = -\left((1.1)(1.5)\sin 128 + (1.8)(1.9)\sin 58 + (1.5)(2.7)\sin 143\right) = -6.64$.

Analytical:

An *x*-*y* frame is positioned at *O*. The *x* and *y* components of all the vectors are determined and denoted as

$$\mathbf{F}_A = \begin{Bmatrix} F_{A_x} \\ F_{A_y} \end{Bmatrix},\ \mathbf{F}_B = \begin{Bmatrix} F_{B_x} \\ F_{B_y} \end{Bmatrix},\ \mathbf{F}_C = \begin{Bmatrix} F_{C_x} \\ F_{C_y} \end{Bmatrix}$$

$$\mathbf{R}_{AO} = \begin{Bmatrix} R_{AO_x} \\ R_{AO_y} \end{Bmatrix},\ \mathbf{R}_{BO} = \begin{Bmatrix} R_{BO_x} \\ R_{BO_y} \end{Bmatrix},\ \mathbf{R}_{CO} = \begin{Bmatrix} R_{CO_x} \\ R_{CO_y} \end{Bmatrix}$$



The moment equation for this body is written as

$$T - R_{AO_y} F_{A_x} + R_{AO_x} F_{A_y} - R_{BO_y} F_{B_x} + R_{BO_x} F_{B_y} - R_{CO_y} F_{C_x} + R_{CO_x} F_{C_y} = 0$$

The angle of position vectors are measured to be $\theta_{R_{AO}} = 208^o$, $\theta_{R_{BO}} = 344^o$ and $\theta_{R_{CO}} = 43^o$. Therefore the position vectors can be expressed in component form as

$$\mathbf{R}_{AO} = \begin{Bmatrix} -0.97 \\ -0.52 \end{Bmatrix},\ \mathbf{R}_{BO} = \begin{Bmatrix} 1.73 \\ -0.50 \end{Bmatrix},\ \mathbf{R}_{CO} = \begin{Bmatrix} 1.10 \\ 1.02 \end{Bmatrix}$$

The applied torque can now be computed as

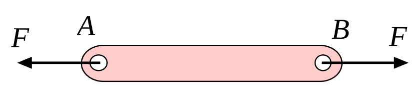
$$T = (-0.52)(1.23) - (-0.97)(-0.86) + (-0.50)(1.46) - (1.73)(1.22) + (1.02)(-2.68) - (1.10)(-0.36) = -6.66$$

Special Cases

If only two or three forces act on a body (and nothing else), and the body is in static equilibrium, the forces are balanced in such a way that they exhibit certain characteristics. We should take advantage of such cases when solving for unknown forces.

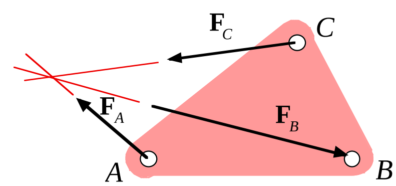
Two-force member

If *only* two forces act on a body that is in static equilibrium, the two forces are along the axis of the link, equal in magnitude, and opposite in direction.

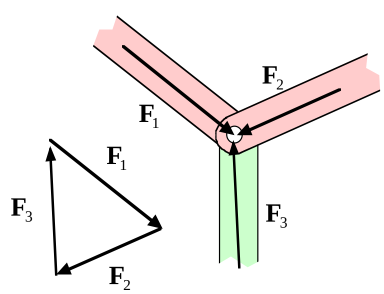


Three-force member

If *only* three forces act on a body that is in static equilibrium, their axes intersect at a single point. This knowledge can help us simplify the solution process in some problems. For example, if the axes of two of the forces are known, the intersection of those two axes can assist us in determining the axis of the third force.



A special case of the three-force member is when three forces meet at a pin joint that is connected between three links. When the system is in static equilibrium, the sum of the three forces must be equal to zero.



Static Equilibrium Analysis of Mechanisms

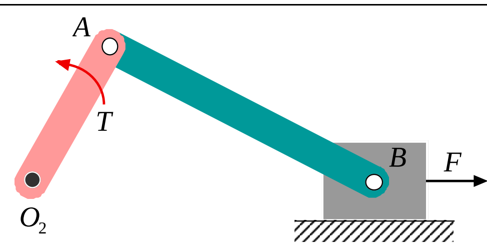
A one degree-of-freedom planar mechanism may contain N moving links. If the mechanism is in static equilibrium, then 3 equilibrium equations can be written for each link, which results into $3N$ equations. In these equations, there are normally $3N$ - 1 unknown reaction force/torque components and 1 unknown applied force or torque. In a static force analysis, the main objective is to solve the equilibrium equations for the unknown applied force/torque. Depending on the method of solution, the reaction forces/torques may also be found in the process. In these notes, two methods for solving the static equilibrium equations are presented: the free-body-diagram (FBD) method, and the power-formula (PF) method.

Free-Body Diagram Method

In this method the static equilibrium equations are constructed based on the free-body diagrams (FBD) of each link. In each FBD, all of the reaction and applied forces/torques that act on that link are considered. For a mechanism of N moving links, the $3N$ equilibrium equations form a set of linear algebraic equations in $3N$ unknowns. Although the equations are linear, due to their large number, a solution by hand may not be practical or may not be the best choice. In such cases, a numerical solution through the use of a computer program is recommended. However, if certain simplifications are made and the number of equations is reduced, then solving the reduced set of equations by hand, either analytically or graphically, may be practical.

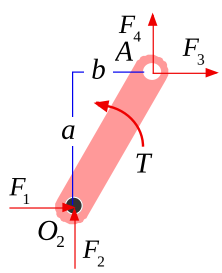
Example 1

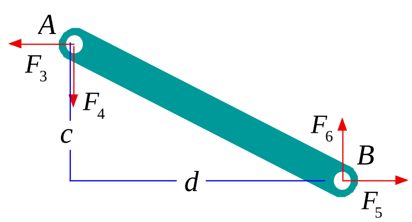
This slider-crank mechanism is in static equilibrium in the shown configuration. A known force **F** acts on the slider block in the direction shown. An unknown torque acts on the crank. Our objective is to determine the magnitude and the direction of this torque in order to keep the system in static equilibrium.

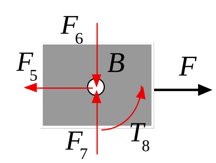


We construct the free body diagrams for each link. The reaction forces at the pin joints are unknown. Each reaction force is described in term of its x and y components, where the direction of each component is assigned arbitrarily. For notational simplification, simple numbered indices are used for all the components. For each link we construct three equilibrium equations.

Link 2: *Link 3:* *Link 4:*







Link 2	Link 3	Link 4
$F_1 + F_3 = 0$	$-F_3 + F_5 = 0$	$-F_5 + F = 0$
$F_2 + F_4 = 0$	$-F_4 + F_6 = 0$	$-F_6 + F_7 = 0$
$-aF_3 + bF_4 + T = 0$	$cF_5 + dF_6 = 0$	$T_8 = 0$
(Sum of moments about O_2)	(Sum of moments about A)	(Sum of moments about B)

The moment arms are measured directly from the figure. Note that F_7 and T_8 are the reaction force and torque due to the sliding joint.

These 9 equations can be put into matrix form. The unknowns and their coefficients are kept on the left-hand side and the only known quantity, the known applied force F is moved to the right-hand side.

This set of 9 equations in 9 unknowns can be solved by any preferred numerical method. If the arbitrarily assigned direction to a force component or a torque is not correct, the obtained solution will be a negative quantity.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & b & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -F \\ 0 \\ 0 \end{Bmatrix}$$

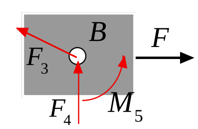
Numerical values for the link lengths are $L_2 = 2.0$ and $L_3 = 4.0$. From the figures we extract the following measurements: $a = 1.8$, $b = 1.0$, $c = 2.0$, $d = 3.6$. Assume the applied force is given to be $F = 10$ units in the negative direction. These values are substituted in the equilibrium equation. The solution to the unknowns is determined to be as shown in array form. The applied torque on the crank is $T = 23.55$ units. As a byproduct of this process, all the reaction forces/torques are also found.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} -10 \\ 5.56 \\ 10 \\ -5.56 \\ 10 \\ -5.56 \\ -5.56 \\ 0 \\ 23.55 \end{Bmatrix}$$

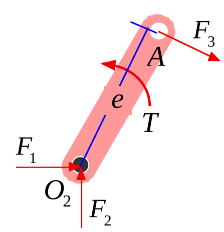
<u>Simplified FBD method</u>

The connecting rod of this mechanism is a two-force member. The reaction forces at A and B must be equal but in opposite directions. These reaction forces are named F_3 and given arbitrary directions.

The FBD of links 2 and 4 can now be constructed. For link 4 we write the sum of forces along the x-axis as $-F_{3(x)} + F = 0$. Since the applied force F is known, $F_{3(x)}$ can be computed. Then based on the angle of F_3, the magnitude of F_3 can be determined.



For link 2 we write the moment equation about O_2 as $-eF_3 + T = 0$. This equation yields the unknown applied torque T.

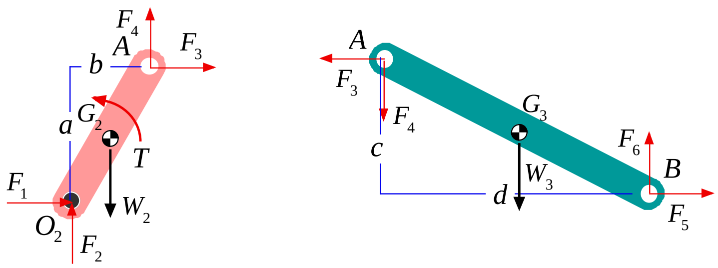


Using numerical values for this example, we have $F_{3(x)} = F = 10$. F_3 makes a 27^o angle with the x-axis, therefore we have $F_3 = F_{3(x)} / \cos 27 = 11.22$. Direct measurement from the figure provides a value for the moment arm as $e = 2.0$, which yields $T = eF_3 = 23.57$.

Example 2

This example is the same slider-crank mechanism from Example 1, where we are asked to include gravitational forces on the links. The mass center for each link is positioned at the geometric center. The free body diagrams for the three links are constructed.

Link 2: *Link 3:* *Link 4:*



$$F_1 + F_3 = 0$$
$$F_2 + F_4 - W_2 = 0$$
$$-aF_3 + bF_4 - \frac{b}{2}W_2 + T = 0$$
(Sum of moments about O_2)

$$-F_3 + F_5 = 0$$
$$-F_4 + F_6 - W_3 = 0$$
$$cF_5 + dF_6 + \frac{d}{2}W_3 = 0$$
(Sum of moments about A)

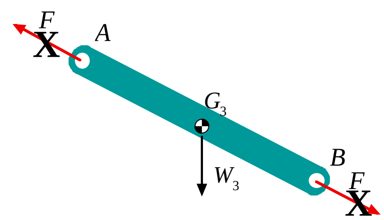
$$-F_5 + F = 0$$
$$-F_6 + F_7 - W_4 = 0$$
$$T_8 = 0$$
(Sum of moments about B)

These 9 equations are expressed in matrix form and solved for the 9 unknowns. We note that the left-hand side of this matrix equation is identical to that in Example 1. The difference is in the right-hand sides.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & b & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ W_2 \\ bW_2/2 \\ \hline 0 \\ W_3 \\ -dW_3/2 \\ \hline -F \\ W_4 \\ 0 \end{Bmatrix}$$

<u>Simplified FBD method!</u>

Since three forces act on the connecting rod, this link is no longer a two-force member. We cannot assume that the reaction forces at A and B are equal, in opposite directions, and along the axis of the link. Therefore, the simplified method is not applicable to this problem. This is the case for most examples when we include gravitational forces.



Coulomb Friction

Coulomb friction can be included between two contacting surfaces in a static force analysis. Given the static coefficient of friction, $\mu^{(s)}$, the friction force can be described as the product of the coefficient of friction and the reaction force normal to the contacting surfaces. The friction force must act in the opposite direction of the tendency of any motion. Since the assumption is that the system is stationary, the tendency of motion must be considered for two cases. The process provides a range of values for the applied load while the system remains in equilibrium.

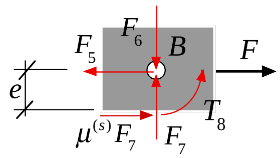
Example 3

We repeat Example 1 with the assumption that dry friction exists between the slider block and the ground. We solve this problem with the FBD method. We may write the complete set of equilibrium equations or take advantage of a 2-force member in the system. Here we construct the complete set of equations. Since the FBD for the crank and the connecting rod are the same as before, we only show the FBD for the slider block.

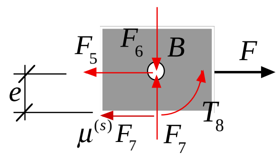
First we assume that the block is about to move to the left. Therefore, the friction force must be directed to the right. Then we assume that the slider block is about to move to the right.

Therefore the friction force must be directed to the left:

$$-F_5 + F + \mu^{(s)} F_7 = 0$$
$$-F_6 + F_7 = 0 \qquad \text{(The block tends to move to the left)}$$
$$T_8 + e\mu^{(s)} F_7 = 0$$



$$-F_5 + F - \mu^{(s)} F_7 = 0$$
$$-F_6 + F_7 = 0 \qquad \text{(The block tends to move to the right)}$$
$$T_8 - e\mu^{(s)} F_7 = 0$$



We construct two complete sets of equations for these two cases. The two sets are presented together as shown. We solve each set of equations twice: once with the positive sign in front of $\mu^{(s)}$ and once with the negative sign. Each solution yields a value for the unknown torque, T_{left} and T_{right}. As long as the applied torque stays in the range of T_{left} to T_{right}, the system remains in static equilibrium.

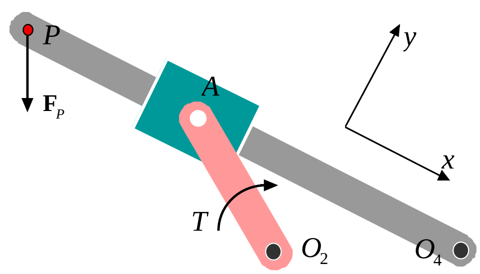
$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & b & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & \pm\mu^{(s)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm e\mu^{(s)} & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -F \\ 0 \\ 0 \end{Bmatrix}$$

Assuming $\mu^{(s)} = 0.25$, $e = 0.5$, and using the numerical values from Example 1, we solve the system of 9 equations in 9 unknowns twice. For the plus sign in front of the terms containing $\mu^{(s)}$, the solution yields $T = 20.68$, where the minus sign in front of those two terms yield $T = 27.35$. Therefore, as long as the applied torque is in the range $20.68 \le T \le 27.35$, the mechanism remains in static equilibrium.

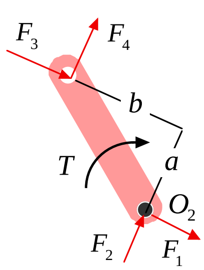
Example 4

In this example, a known force acts at point P an unknown torque acts on the crank. It is assumed that dry friction exists at the sliding joint.

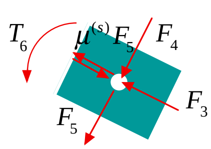
To solve this problem, we construct FBD's for the links. Although it is not necessary, in order to simplify the process of projecting the reaction forces unto the x-y axes, the x-y axes are rotated in such a way that the x-axis is along the axis of link *(4)*.



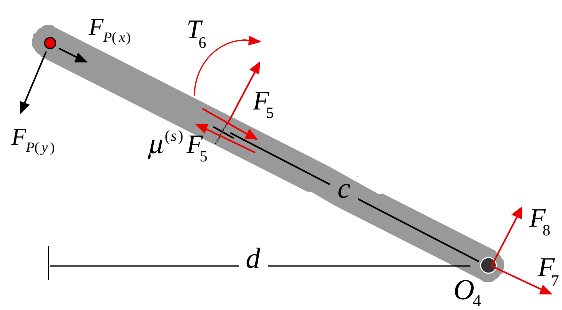
Link 2:



Link 3:



Link 4:



$F_1 + F_3 = 0$ $\qquad -F_3 \pm \mu^{(s)} F_5 = 0$ $\qquad F_7 \mp \mu^{(s)} F_5 + F_{P(x)} = 0$

$F_2 + F_4 = 0$ $\qquad -F_4 - F_6 = 0$ $\qquad F_5 + F_8 - F_{P(y)} = 0$

$-aF_3 - bF_4 - T = 0$ $\qquad T_6 = 0$ $\qquad -T_6 - cF_5 + dF_P = 0$

These equations can be expressed in matrix form and solved twice to find the range of values for the applied torque T.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & -b & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & -1 & 0 & \pm\mu^{(s)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mp\mu^{(s)} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -c & -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ T_6 \\ F_7 \\ F_8 \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline -F_{P(x)} \\ F_{P(y)} \\ -dF_P \end{Bmatrix}$$

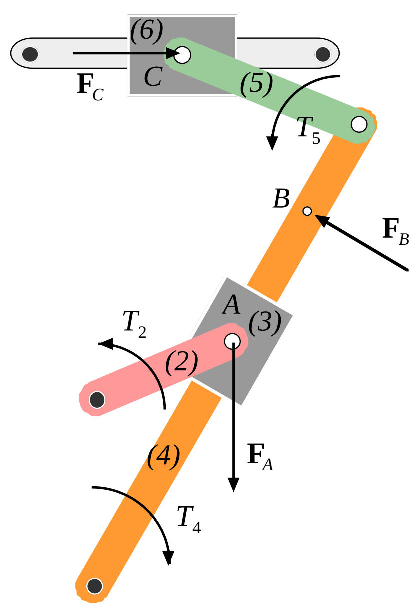
Power Formula Method

The power formula method constructs *one equation* in *one unknown* for a one degree-of-freedom mechanism regardless of its number of links and joints. Only applied forces and torques, including the unknown applied force or toque, appear in the equation. This means that the reaction forces and torques have been eliminated from the power formula.

In order to use the power formula, although the system is in static equilibrium, we first assume that the system is in motion. We assign an arbitrary velocity (angular or linear) to one of the links; e.g., the rotation of the crank, and determine the velocities that are needed in the power formula. These are the velocities of the points were the applied forces act on, and the angular velocity of the links that a torques are applied to.

The equation for the power formula is:

$$\sum \mathbf{F}_P \cdot \mathbf{V}_P + \sum T\,\omega = 0 \qquad \text{(spf.1)}$$



In this equation $\mathbf{F}_P$ is a typical applied force acting on a link at point P, $\mathbf{V}_P$ is the velocity (absolute) of point P, T is a typical applied torque acting on a link, and ω is the angular velocity of that link. If the torque and the angular velocity are in the same direction, the product $T\omega$ is positive, otherwise the product is negative.

As an example consider the six-bar mechanism shown. Three applied forces and three applied torques act on the system. We need to find the *imaginary* velocities of points A, B, and C, and the *imaginary* angular velocities of links 2, 4, and 5. We can find these velocities by assigning an arbitrary value to the velocity of one of the degrees-of-freedom, say to the angular velocity of link 2. We may apply any preferred method to solve for the other velocities. The value of this arbitrarily assigned velocity or its direction will not change the result obtained from the power formula.

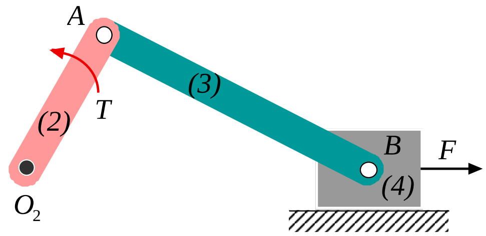
Note: This method is also referred to as the *energy* method or the *virtual work* method. Since we have not discussed the concepts of virtual work and virtual displacements, the method is presented using velocities although the system is in static equilibrium.

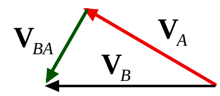
Note: Since reaction forces and torques do not appear in the power formula, this method **is not** applicable when Coulomb friction is present.

Example 5

We apply the power formula to the slider-crank mechanism of Example 1. One external torque and one external force act on this system. We need to determine the imaginary angular velocity of link 2 and the velocity of the slider block.

We assume link 2 rotates with an angular velocity $\omega_2 = 1$ rad/sec, CCW. We construct the velocity polygon and determine the velocity of the slider block (point *B*). Considering the length of the crank as $L_2 = 2.0$, the polygon yields $V_B = 2.4$.





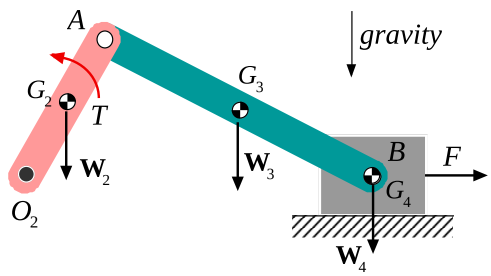
For this problem the power formula becomes $\mathbf{F} \cdot \mathbf{V}_B + T\omega_2 = 0$. Since **F** and $\mathbf{V}_B$ are along the same axis but in opposite directions, the formula simplifies to $-FV_B + T\omega_2 = 0$, or $T = FV_B / \omega_2 = (10)(2.4)/1.0 = 24.0$.

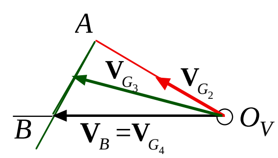
Example 6

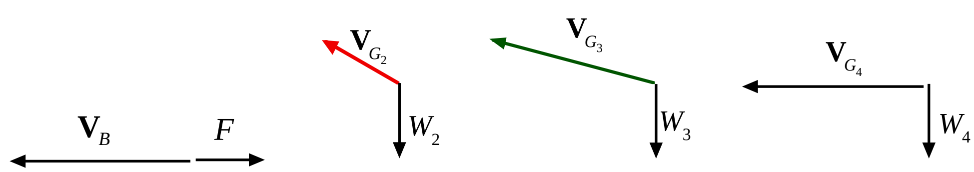
We apply the power formula to the problem of Example 2. We assume an imaginary $\omega_2 = 1$ rad/sec, CCW, and construct the velocity (as in the previous example). The velocities of the points where the applied forces act, including the gravitational forces, are determined. The PF is then expressed as:

$$\mathbf{F} \cdot \mathbf{V}_B + \mathbf{W}_2 \cdot \mathbf{V}_{G_2} + \mathbf{W}_3 \cdot \mathbf{V}_{G_3} + \mathbf{W}_4 \cdot \mathbf{V}_{G_4} + T\omega_2 = 0$$

The dot product between a velocity vector and a force vector can be obtained either analytically or graphically (the angle between the two vectors can be measured directly from the figure). The PF yields the value for the unknown applied torque.







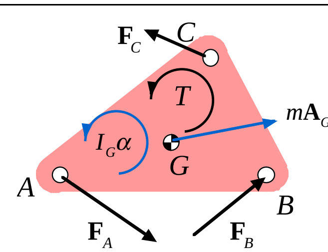
Dynamic Force Analysis

Dynamic Equilibrium Equations

The dynamic equilibrium equations for a single body in planar motion are the revised version of the static equilibrium equations, expressed as

$$\sum \mathbf{F}_i = m\mathbf{A}_G \quad \Rightarrow \quad \begin{cases} \sum F_{i(x)} = mA_{G(x)} \\ \sum F_{i(y)} = mA_{G(y)} \end{cases} \qquad \text{(de.1)}$$

$$\sum T_j + \sum M_{i(G)} = I_G \alpha \qquad \text{(de.2)}$$



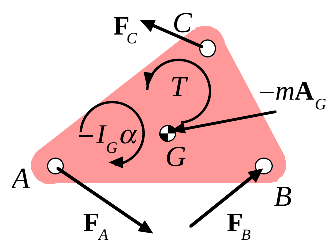
The force equilibrium of Eq. (de.1) states that the sum of forces that act on a body must be equal to the mass of the body, *m*, times the acceleration of the mass center, $\mathbf{A}_G$. This equation is derived from Newton's second Law of motion. The moment equilibrium of Eq. (de.2) states that the sum of all the applied toques and the moments of all forces that act on a body with respect to its mass center must be equal to the body's moment of inertia, I_G, times the angular acceleration of the body, α. The moment of inertia is defined with respect to an axis passing through the mass center and perpendicular to the plane. Equations (de.1) and (de.2) are also known as the translational and rotational equations of motion respectively.

D'Alembert's Principle

The D'Alembert's principle considers a rearranged form of the dynamic equilibrium equations as

$$\sum \mathbf{F}_i - m\mathbf{A}_G = \mathbf{0} \qquad \text{(de.3)}$$

$$\sum T_j + \sum M_{i(G)} - I_G\,\alpha = 0 \qquad \text{(de.4)}$$



Although this rearrangement may appear trivial, it has profound implications. One interpretation of this principle is that if $-m\mathbf{A}_G$ and $-I_G\,\alpha$ are viewed as a force and a torque respectively, then the dynamic equilibrium equations become identical to the static equilibrium equations. The terms $-m\mathbf{A}_G$ and $-I_G\,\alpha$ are called *inertial force* and *inertial torque*.

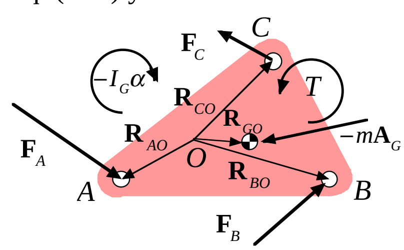
Revised Rotational Equation

D'Alembert's principle provides us a simple set of steps to revise Eq. (de.2) if the sum of moments is taken about a point that is not the mass center. If we consider $-m\mathbf{A}_G$ and $-I_G\,\alpha$ as additional force and torque that act on the body, and consider point O as the reference point instead of the mass center, the rotational static equilibrium of Eq. (se.2) yields

$$\sum T_j + \sum M_{i(O)} - I_G\,\alpha - m\mathbf{R}_{GO} \times \mathbf{A}_G = 0$$

or,

$$\sum T_j + \sum M_{i(O)} = I_G\,\alpha + m\mathbf{R}_{GO} \times \mathbf{A}_G \qquad \text{(de.5)}$$



This equation clearly shows that if the origin is not the mass center, then we must consider the moment caused by the inertia force in the moment equation.

Dynamic Equilibrium Analysis of Mechanisms

The process of dynamic force analysis of mechanisms is only slightly different from that of the static force analysis, especially if we look at the dynamic equilibrium equations from the perspective of D'Alembert's principle. The main difference is that in a dynamic force analysis we need to include the linear and angular accelerations of each link in the process of solving the dynamic equilibrium equations.

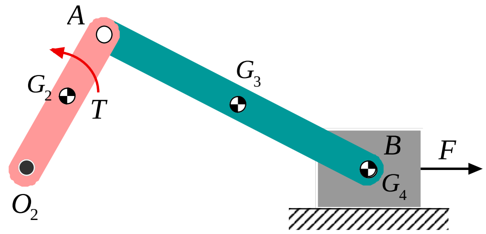
For the dynamic force analysis, as in the static force analysis, we consider the free-body diagram (FBD) method and the power formula (PF) method.

Free-Body Diagram Method

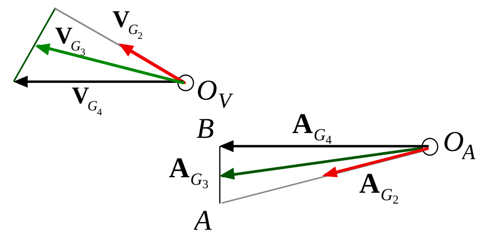
The dynamic force analysis requires incorporating the applied and reaction forces and torques, and the inertial force and torque, in the FBD of each link. Inclusion of the applied and reaction forces and torques in this process is identical to that of the static force analysis. In order to include the inertial forces and torques in the process, we need to know the linear and angular accelerations for each link.

Example 7

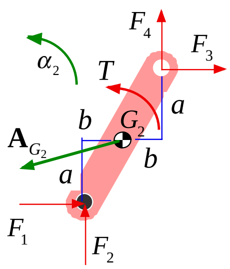
The mass and moment of inertia for the links of this slider-crank are given. A known force ***F*** acts on the slider block, and an unknown torque T acts on the crank. In the depicted configuration, the angular velocity and acceleration of the crank are given. The objective is to find the magnitude and the direction of the unknown torque.



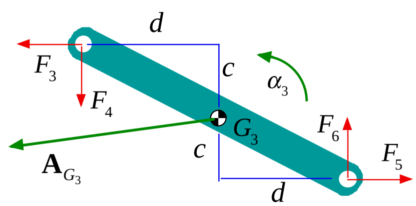
Based on the given angular velocity and acceleration of the crank, polygons are constructed and the angular velocity and acceleration of all the links are found (magnitudes and directions—ω_2, α_2 and α_3 are CCW, and ω_3 is CW). Then, in a process similar to that of the static equilibrium (refer to Example 1), the FBD for each link is constructed.



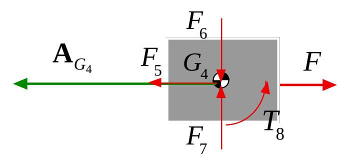
Link 2:



Link 3:



Link 4:



$$F_1 + F_3 = m_2 A_{G_2(x)} \qquad -F_3 + F_5 = m_3 A_{G_3(x)} \qquad -F_5 + F = m_4 A_{G_4}$$

$$F_2 + F_4 = m_2 A_{G_2(y)} \qquad -F_4 + F_6 = m_3 A_{G_3(y)} \qquad -F_6 + F_7 = 0$$

$$aF_1 - bF_2 - aF_3 + bF_4 + T = I_{G_2}\alpha_2 \qquad cF_3 + dF_4 + cF_5 + dF_6 = I_{G_3}\alpha_3 \qquad T_8 = 0$$

Note that $A_{G_2(x)}$, $A_{G_2(y)}$, $A_{G_3(x)}$, $A_{G_3(y)}$ and A_{G_4} are negative, where α_2 and α_3 are positive. These equations are put into matrix form. This set of 9 equations in 9 unknowns can be solved by any preferred numerical method.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & -b & -a & b & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & c & d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G_2(x)} \\ m_2 A_{G_2(y)} \\ I_{G_2}\alpha_2 \\ \hline m_3 A_{G_3(x)} \\ m_3 A_{G_3(y)} \\ I_{G_3}\alpha_3 \\ \hline m_4 A_{G_4} - F \\ 0 \\ 0 \end{Bmatrix}$$

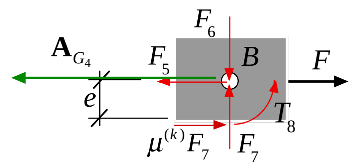
Coulomb Friction

Coulomb friction can be included between two contacting surfaces in a dynamic force analysis. Given the dynamic (kinetic) coefficient of friction, $\mu^{(k)}$, the friction force can be described as the product of the coefficient of friction and the reaction force normal to the contacting surfaces. The friction force must act in the opposite direction of the motion. The process is illustrated through a simple example.

Example 8

We consider the slider-crank mechanism of Example 7 with the assumption that dry friction exists between the slider block and the ground. We construct the complete set of equations for this problem. Since the FBD for the crank and the connecting rod are the same as in Example 7, we only show the FBD for the slider block.

Since, according to the velocity polygon the block is moving to the left, the friction force must be directed to the right. The equilibrium equations for the block are:



$$-F_5 + F + \mu^{(k)} F_7 = m_4 A_{G_4}$$

$$-F_6 + F_7 = 0$$

$$T_8 + e\mu^{(k)} F_7 = 0$$

The complete set of equations can now be presented in matrix form. The solution to this set of equations yields all the unknowns.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ a & -b & -a & b & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & d & c & d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & \mu^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e\mu^{(k)} & 1 & 0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ T_8 \\ T \end{Bmatrix} = \begin{Bmatrix} m_2 A_{G_2(x)} \\ m_2 A_{G_2(y)} \\ I_{G_2}\alpha_2 \\ \hline m_3 A_{G_3(x)} \\ m_3 A_{G_3(y)} \\ I_{G_3}\alpha_3 \\ \hline m_4 A_{G_4} - F \\ 0 \\ 0 \end{Bmatrix}$$

Power Formula Method

The power formula (PF) for dynamic force analysis is a revised form of the formula from the static force analysis. If we consider the inertial forces and torques as any other applied forces and torques, the revised formula can be expressed as

$$\sum \mathbf{F}_P \cdot \mathbf{V}_P + \sum T\,\omega = \sum m_i \mathbf{A}_{G_i} \cdot \mathbf{V}_{G_i} + \sum I_{G_i} \alpha_i\, \omega_i \qquad \text{(dpf.1)}$$

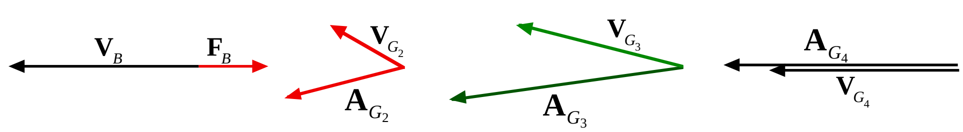
In this formula, the velocities and accelerations are actual and not imaginary. If the angular velocity and acceleration are in the same direction, the product $\alpha\,\omega$ is positive, otherwise the product is negative.

Example 9

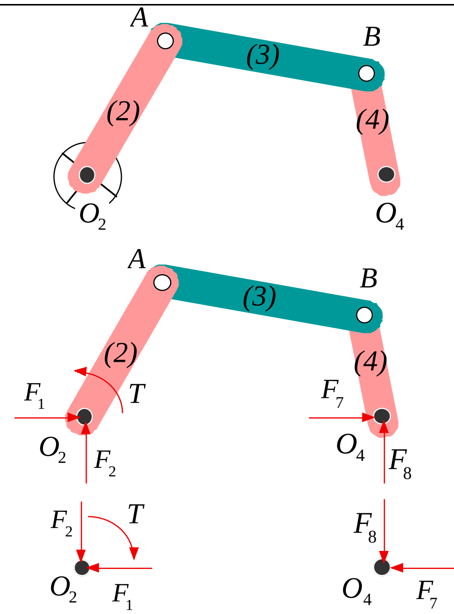
We apply the PF method to the slider-crank of Example 8. The formula for this system can be expressed as

$$\mathbf{F} \cdot \mathbf{V}_B + T\omega_2 = m_2 \mathbf{A}_{G_2} \cdot \mathbf{V}_{G_2} + m_3 \mathbf{A}_{G_3} \cdot \mathbf{V}_{G_3} + m_4 \mathbf{A}_{G_4} \cdot \mathbf{V}_{G_4} + I_{G_2}\alpha_2\omega_2 + I_{G_3}\alpha_3\omega_3 + I_{G_4}\alpha_4\omega_4$$

In addition to the angular velocities and accelerations, from the velocity and acceleration polygons, the velocity and acceleration of the mass centers are determined. All the known quantities are substituted in the formula to find the unknown applied torque *T*.



Shaking Force and Shaking Torque/Moment



Dynamic forces that act on a mechanism cause the foundation (the ground) to shake. The *shaking force* is defined as the sum of reaction forces the links of a mechanism apply on the ground link. For a four-bar mechanism, since there are only two links that are directly pinned to the ground at O_2 and O_4, the sum of reaction forces acting on the ground at these two points is the shaking force denoted as

$$\mathbf{F}_s = -\begin{Bmatrix} F_1 + F_7 \\ F_2 + F_8 \end{Bmatrix}$$

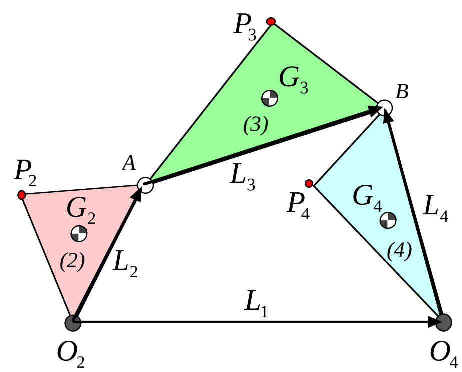
where F_1, F_2, F_7 and F_8 could be positive or negative.

When link *2* is the input link (the crank), a motor about the axis of O_2 must rotate it. The motor applies a torque T on link *2* and an opposite torque on the ground. The torque that is applied by the motor to the ground is called the *shaking torque (moment)*:

$$T_s = -T$$

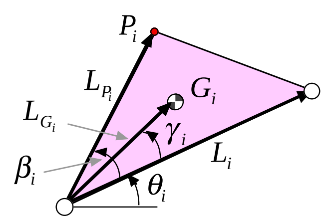
The shaking force tends to move the ground (the frame) up and down, and left and right. The shaking torque tends to rock the frame about the vertical axis to the plane.

A Matlab Program



A Matlab program (**`fourbar_force.m`**) for static and dynamic force analysis of a four-bar mechanism is provided. The four-bar is considered in its most general form as shown. The program allows one known force and one known torque to be applied on each link. The gravitational force may also be considered to act in the negative *y*-direction on the system. The program requires only one unknown torque, which could be considered on only one of the links.

For the purpose of generality, the mass center of a link, G_i, is positioned from the reference point of the link by an angle γ_i and a length L_{G_i}. The force application point, P_i, is positioned by an angle β_i and a length L_{P_i}. Note that the reference point for link *2* is O_2, for link *3* is *A*, and for link *4* is O_4.



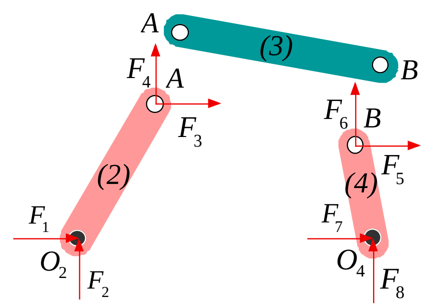
This program retrieves the data from a file named **`fourbar_force_data.m`**. The user is required to provide the following data in this file:

- Constant values for the link lengths (L_1, L_2, L_3, L_4)
- Angle of the crank (θ_2)
- Estimates for the angles of the coupler and the follower (θ_3, θ_4)
- Angular velocity and acceleration of the crank (ω_2, α_2)
- Constant values for the position of *P* points ($L_{P_2O_2}$, L_{P_3A}, $L_{P_4O_4}$, β_2, β_3, β_4)

- Constant values for the position of G points ($L_{G_2O_2}$, L_{G_3A} , $L_{G_4O_4}$, γ_2 , γ_3 , γ_4)
- Known applied forces ($\mathbf{F}_{P_2}$, $\mathbf{F}_{P_3}$, $\mathbf{F}_{P_4}$)
- Known applied torques (T_2 , T_3 , T_4)
- Masses and moments of inertia (m_2 , m_3 , m_4 , I_2 , I_3 , I_4)
- Gravitational constant (g) (the value of g determines the system of units for the analysis)

The program prompts the user for the following information:

- Should the gravitational force be included?
    - Answer y for yes or n for no
- The unknown torque is applied to which link?
    - Answer 2, 3 or 4
- Static or dynamic force analysis?
    - Answer s for static or d for dynamic



The program reports the results for the reaction forces at O_2 and A on link 2, at B and O_4 on link 4, and the unknown applied torque. The reaction forces acting at A and B on link *3* could easily be determined by the user. The program also reports the shaking force and the shaking torque.