Rolling Disk

Deriving constraint equations for a thin disk rolling on the ground is more interesting than other types of constraints that we have developed so far. The difficulty here is that any point on the disk circumference can become in contact to any point on the ground surface. Therefore we cannot specify any particular point a priori as a contact candidate; i.e., for any given set of coordinates of the disk, the contact point should be identified.

We assume the ground is defined as \( z = 0 \) and the normal unit vector to the ground surface is \( \mathbf{z} \). We further assume that the center of the disk is positioned by vector \( \mathbf{r} \), and the orientation of the disk is described by matrix \( \mathbf{A} \). We also assume that the unit vector \( \mathbf{\eta} \) defines the normal to the disk surface, where it represents the second column of \( \mathbf{A} \); i.e., \( \mathbf{\eta} = \begin{bmatrix} a_{12} & a_{22} & a_{32} \end{bmatrix}' \).

The instantaneous point of contact on the disk is named point \( C \) and the contact point on the ground is named point \( G \). The vector connecting the center to \( C \) is \( \mathbf{s}_C \), and the vector connecting the center to \( G \) is \( \mathbf{d} \). It is important to realize that although \( \mathbf{s}_C = \mathbf{d} \), \( \mathbf{s}_C \neq \mathbf{d} \) unless the contact point is not in motion. Our first objective is to construct vector \( \mathbf{d} \) (or \( \mathbf{s}_C \)).

The intersection of the ground and the plane of the disk can be described by vector \( \mathbf{t} \) as

\[
\mathbf{t} = \mathbf{\eta z} = \begin{bmatrix} a_{22} & -a_{12} & 0 \end{bmatrix}'
\]

A vector in the plane of the ground normal to the intersect line can be obtained as

\[
\mathbf{n} = \mathbf{t z} = \begin{bmatrix} a_{12} & a_{22} & 0 \end{bmatrix}'
\]

Note that \( \mathbf{n} \) is the projection of \( \mathbf{\eta} \) onto the x-y plane. A vector along \( \mathbf{d} \) can be constructed as \( \mathbf{\eta t} \) which has a magnitude of \( \sqrt{1-a_{32}^2} \). We then construct vector \( \mathbf{d} \) as

\[
\mathbf{d} = \frac{R}{\sqrt{1-a_{32}^2}} \mathbf{\eta t} = \frac{R}{\sqrt{1-a_{32}^2}} \begin{bmatrix} a_{12}a_{32} \\ a_{22}a_{32} \\ 1-a_{32}^2 \end{bmatrix} \quad (1)
\]

The z-component of this vector set equal to the z-component of \( \mathbf{r} \) provides the constraint to keep the disk in point-contact with the ground; i.e.,

\[
z - R\sqrt{1-a_{32}^2} = 0 \quad (2)
\]

If we look at the disk from the side, we note that the angle that the plane of the disk makes with the ground is the same as the angle between \( \mathbf{z} \) and \( \mathbf{\eta} \). Therefore, \( a_{32} \) represents \( \cos \theta \), and \( \sin \theta \) can be computed as \( \sqrt{1-a_{32}^2} \). Hence, \( R\sqrt{1-a_{32}^2} \) must be equal to the z-coordinate of the disk, which is our constraint.

Now we can impose additional constraints on the contact point. Assume that we do not want the contact point to slip as the disk rolls. The no-slip condition is imposed as a velocity constraint as \( \mathbf{\dot{r}}_C = 0 \). Since \( \mathbf{r}_C = \mathbf{r} + \mathbf{s}_C \) and the velocity of \( C \) is zero, the velocity constraint can be expressed as \( \mathbf{\dot{r}} - \mathbf{s}_C \mathbf{\omega} = 0 \), or

\[
\begin{bmatrix} 1 & -\mathbf{d} \end{bmatrix} \begin{bmatrix} \mathbf{\dot{r}} \\ \mathbf{\omega} \end{bmatrix} = 0 \quad (3)
\]
We must note that since the wheel is allowed to roll, \( \mathbf{R} \neq 0 \). Therefore, in order to obtain the acceleration constraint, we simply take the time derivative of Eq. 3:

\[
\begin{bmatrix}
1 & -\mathbf{d}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{r}}
\end{bmatrix}_{\omega} = \mathbf{d} \omega
\]  

(4)

where \( \dot{\mathbf{d}} \) must be computed as the time derivative of Eq. 1. The time derivatives of the vectors that are involved in this computation are obtained as

\[
\begin{align*}
\dot{\eta} &= \{\dot{a}_{12}, \dot{a}_{22}, \dot{a}_{32}\}' = \tilde{\omega} \eta \\
\dot{\mathbf{t}} &= \{\dot{a}_{22}, -\dot{a}_{12}, 0\}' \\
\dot{\mathbf{n}} &= \{\dot{a}_{12}, \dot{a}_{22}, 0\}'
\end{align*}
\]

Hence \( \dot{\mathbf{d}} \) is obtained as

\[
\dot{\mathbf{d}} = \frac{a_{32} \dot{a}_{32}}{1 - a_{32}^2} \mathbf{d} + \frac{R}{\sqrt{1 - a_{32}^2}} (\dot{\eta} \mathbf{t} - \dot{\mathbf{n}} \eta)
\]

(5)

The first and second rows of Eq. 3 (or Eq. 4) provide the velocity (or acceleration) constraints for the no-slip condition. These are non-holonomic constraints; i.e., there are no constraints for this condition at the coordinate level. The third rows of these equations provide the time derivatives of the contact point in the \( z \)-direction, which is a holonomic constraint as presented in Eq. 2.

The no-slip condition in Eq. 3 (or Eq. 4) is enforced at the contact point in the \( x-y \) plane. However, often it is necessary to treat the no-slip condition in the longitudinal and lateral directions separately. In the longitudinal direction, Eq. 3 is projected onto the \( t \)-axis as

\[
\begin{bmatrix}
\dot{\mathbf{t}}' & -t' \dot{\mathbf{d}}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{r}}
\end{bmatrix}_{\omega} = 0
\]

where its time derivative becomes

\[
\begin{bmatrix}
\dot{\mathbf{t}}' & -t' \dot{\mathbf{d}}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{r}}
\end{bmatrix}_{\omega} = t' \dot{\mathbf{d}} \omega - t' (\dot{\mathbf{r}} - \dot{\mathbf{d}} \omega)
\]

Similarly, for the lateral no-slip condition we have

\[
\begin{bmatrix}
\dot{\mathbf{n}}' & -n' \dot{\mathbf{d}}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{r}}
\end{bmatrix}_{\omega} = 0
\]

where its time derivative becomes

\[
\begin{bmatrix}
\dot{\mathbf{n}}' & -n' \dot{\mathbf{d}}
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{r}}
\end{bmatrix}_{\omega} = n' \dot{\mathbf{d}} \omega - n' (\dot{\mathbf{r}} - \dot{\mathbf{d}} \omega)
\]

Depending on the application, the no-slip condition can be enforced, for example, in the lateral direction but not in the longitudinal direction.\(^1\)

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\(^1\) If we project Eq. 3 onto the \( z \)-axis, we simply extract the third row of the equation, which is the time derivative of Eq. 2.