Lesson 14: Reaction Forces

Array of Reaction Forces and Moments

- The constrained equations of motion

\[ M \ddot{\mathbf{v}} = \mathbf{g} + \mathbf{g}^{(c)} \]

\[ (c) \mathbf{g} = \mathbf{D}^T \lambda \]

- The forces and moments that appear in \((c) \mathbf{g}\) are the reaction forces and moments
  - A reaction moment could be (but not necessarily) the result of a reaction force having a moment arm with respect to the mass center.
  - We examine the reaction forces/moments for several typical constraints

Spherical Joint

- A spherical joint between bodies \(i\) and \(j\):
  - Jacobian matrix

\[
\mathbf{D}^{(s,3)}_{(3 \times 12)} = \begin{bmatrix} -\mathbf{I} & \hat{s}_i^p & \mathbf{I} & -\hat{s}_j^p \end{bmatrix}
\]

  - Array of reaction forces/moments

\[
\mathbf{D}^T \lambda = \begin{bmatrix} -\mathbf{I} & -\hat{s}_i^p & \mathbf{I} & -\hat{s}_j^p \end{bmatrix} \lambda
\]

  - Interpretation: \(\lambda\) is a 3-array representing exactly the reaction force acting at \(P_i\). The same force but in the opposite direction acts at \(P_j\). A spherical joint does not produce reaction torques. However, when a reaction force is moved to the corresponding mass center, the moment associated with that force must be included in the rotational equations of motion. These reaction moments are automatically taken care of by the Jacobian matrix.

Spherical-Spherical Joint

- A spherical-spherical joint between bodies \(i\) and \(j\):
  - Jacobian matrix

\[
\mathbf{D}^{(s+1)}_{(1 \times 12)} = \begin{bmatrix} -\mathbf{d}^T & \mathbf{d}^T \hat{s}_i^p & \mathbf{d}^T & -\mathbf{d}^T \hat{s}_j^p \end{bmatrix}
\]

  - Array of reaction forces/moments
\[
D^\top \lambda = \begin{bmatrix}
-d \\
-s_i^p d \\
-s_j^p d \\
\end{bmatrix} \lambda = \begin{bmatrix}
-I \\
-s_i^p I \\
-s_j^p I \\
\end{bmatrix} d \lambda
\]

Interpretation: This joint contains a single Lagrange multiplier, which its value is proportional to the magnitude of the reaction force. The reaction force is \( d \lambda \) which acts exactly along the axis of the link. If we define a unit vector \( u \) along \( d \lambda \), the reaction force can be represented as \( \ell u \lambda \), where \( \ell \) is the length of the link. This means that the magnitude of the reaction force is \( \ell \lambda \). This joint does not produce reaction torques. The reaction moments are the result of the reaction forces having moment arms with respect to their corresponding mass centers.

Two Perpendicular Vectors (n1)

- Two vectors fixed to bodies \( i \) and \( j \) must remain perpendicular:
  We assume that the two vectors, \( u_i \) and \( u_j \), are unit vectors.

  - Jacobian matrix
    \[
    D^{(n1)} = \begin{bmatrix}
    0 & -u_i^\top u_j & 0 & -u_i^\top u_j \\
    u_i & u_i & 0 & u_j \\
    0 & 0 & -u_i^\top u_j & 0 \\
    \end{bmatrix}
    \]

  - Array of reaction forces/moments
    \[
    D^\top \lambda = \begin{bmatrix}
    0 \\
    \lambda \\
    \end{bmatrix} = \begin{bmatrix}
    \lambda \\
    \lambda \\
    \end{bmatrix}
    \]

    where \( u = u_i u_j \)

  - Interpretation: The unit vector \( u \) is perpendicular to the original two unit vectors about which the reaction torques act in opposite directions on the two bodies. The Lagrange multiplier \( \lambda \) is exactly the magnitude of the reaction moment. Note that there are no reaction forces associated with this constraint.

Two perpendicular vectors (n2)

- Two perpendicular vectors; one is fixed to body \( i \) and one connects body \( i \) to body \( j \):

  - Jacobian matrix
    \[
    D^{(n2)} = \begin{bmatrix}
    -s_i^T \left( d + s_j^p \right) s_i^T & -s_i^T s_j^p \\
    \end{bmatrix}
    \]

  - Array of reaction forces/moments
    \[
    \lambda = \begin{bmatrix}
    0 \\
    0 \\
    \end{bmatrix}
    \]

    where \( u = \lambda(\tilde{d} + \tilde{s}_j^p) \)}
\[ \mathbf{D}^T \lambda = \begin{bmatrix} -s_i \\ -(\tilde{\mathbf{d}} + \tilde{s}_j^p) s_i \\ s_i \\ \tilde{s}_j^p \end{bmatrix} \lambda = \begin{bmatrix} -s_i \\ -\tilde{\mathbf{d}}_{i,j} s_i \\ \mathbf{I} \\ \tilde{s}_j^p \end{bmatrix} \lambda = \begin{bmatrix} -\mathbf{I} \\ -\tilde{\mathbf{d}}_{i,j} \end{bmatrix} s_i \lambda \]

Interpretation: This joint contains a single Lagrange multiplier. The reaction force is \( \mathbf{s}_i \lambda \) which acts exactly along the axis of vector \( \mathbf{s}_i \) on the two bodies. On body \( j \) the reaction force acts at point \( P_j \), where on body \( i \) the reaction force acts at an imaginary point on body \( i \) that coincides with \( P_j \). The reaction moments are the result of the reaction forces having very different moment arms with respect to their corresponding mass centers.

Other Constraints

- Similar interpretation can be used for other types of constraints.