Lesson 12: Equations of Motion

Newton’s Laws

- **First Law:** A particle remains at rest or continues to move in a straight line with constant speed if there is no force acting on it
- **Second Law:** The acceleration of a particle is proportional to and in the same direction of the force acting on it
- **Third Law:** The forces of action and reaction between interacting particles are equal in magnitude, collinear, and opposite in direction

Dynamics of A Particle

- A particle (point)
  - Mass \( m^{(p)} \)
  - Force vector \( \vec{f} \)
  - Acceleration vector \( \vec{a} \) or \( \vec{a} \)
  - Equation of motion \( m^{(p)} \vec{a} = \vec{f} \)

- Particle \( i \)
  - Equation of motion
    \[ m^{(i)} \vec{a}^{(i)} = \vec{f}^{(i)} \text{ or } m^{(i)} \vec{a}^{(i)} = \vec{f}^{(i)} \]
  - In expanded form
    \[
    \begin{bmatrix}
      m^{(i)} & 0 & 0 \\
      0 & m^{(i)} & 0 \\
      0 & 0 & m^{(i)}
    \end{bmatrix}
    \begin{bmatrix}
      \vec{x}^{(i)} \\
      \vec{y}^{(i)} \\
      \vec{z}^{(i)}
    \end{bmatrix}
    =
    \begin{bmatrix}
      \vec{f}_{(x)}^{(i)} \\
      \vec{f}_{(y)}^{(i)} \\
      \vec{f}_{(z)}^{(i)}
    \end{bmatrix}
    
    
Dynamics of A System of Particles

- A system of \( p \) particles
  - Center of mass (centroid) is positioned with vector \( \vec{r} \)
  - Equation of motion for the centroid: \( m \vec{a} = \vec{f} \)

where \( m = \sum_{i=1}^{p} m^{(i)} \) and \( \vec{f} = \sum_{i=1}^{p} \vec{f}^{(i)} \)

- Follow the derivation of the equations of motion for the mass center of a system of particles in Sec. 8.2. Observe how the internal reaction forces between the particles cancel each other!

- Position of the center of mass
  \[ \vec{r} = \frac{1}{m} \sum_{i=1}^{p} m^{(i)} \vec{r}^{(i)} \]

- Resultant equation
  \[ \sum_{i=1}^{p} m^{(i)} \vec{s}^{(i)} = \vec{0} \]
Note that \( \mathbf{r}^i \) locates particle \( i \) from the origin of the reference frame where \( \mathbf{s}' \) locates the particle from the mass center!

**Translational Equations of Motion for A Body**

- Translational equations of motion (centroidal) for a body are
  \[ m \ddot{\mathbf{r}} = \mathbf{f} \]

  In expanded form
  \[
  \begin{bmatrix}
  m & 0 & 0 \\
  0 & m & 0 \\
  0 & 0 & m
  \end{bmatrix}
  \begin{bmatrix}
  \ddot{x} \\
  \ddot{y} \\
  \ddot{z}
  \end{bmatrix} =
  \begin{bmatrix}
  f_{(x)} \\
  f_{(y)} \\
  f_{(z)}
  \end{bmatrix}
  \]

- Mass of the body: \( m \)
- Sum of forces acting on the body: \( \mathbf{f} \)
- Acceleration of the mass center: \( \ddot{\mathbf{r}} \)

**Moment of A Force**

- Moment of a force acting on a body at point \( P \)
  is computed as
  \[ \mathbf{n} = \hat{\mathbf{s}}^P \mathbf{f} \]

  where \( \hat{\mathbf{s}}^P \) locates point \( P \) from the mass center of the body

**Rotational Equations of Motion for A Body**

- The derivation shown here is much simpler than the one given in the textbook!
  - Assume that the body is made of infinite \( (p) \) number of particles
  - For particle \( i \) the equation of motion is
    \[ m^i \ddot{\mathbf{r}}^i = \mathbf{f}^i + \sum_{j=1}^{p} \mathbf{f}^{i,j} \]

    where,
    - \( \mathbf{f}^i \) is an external force that may exist for some of the particles
    - \( \mathbf{f}^{i,j} ; j = 1, ..., p \) are the reaction forces exerted on this particle by other particles
  - Pre-multiply this equation by \( \hat{\mathbf{s}}^i \):
    \[ m^i \hat{\mathbf{s}}^i \ddot{\mathbf{r}}^i = \hat{\mathbf{s}}^i \mathbf{f}^i + \hat{\mathbf{s}}^i \sum_{j=1}^{p} \mathbf{f}^{i,j} \]
    \[ \Rightarrow m^i \hat{\mathbf{s}}^i (\ddot{\mathbf{r}} + \ddot{s}^i) = \hat{\mathbf{s}}^i \mathbf{f}^i + \hat{\mathbf{s}}^i \sum_{j=1}^{p} \mathbf{f}^{i,j} \]
  - We write this equation for every particle and then sum over all the particles:
\[ \sum_{i=1}^{p} m^i s^i \ddot{r} + \sum_{i=1}^{p} m^i \ddot{s}^i s^i = \sum_{i=1}^{p} \ddot{s}^i f^i + \sum_{i=1}^{p} \dddot{s}^i \sum_{j=1}^{p} f^{i,j} \]  

We examine each of the four terms separately!

(a) In this term \( \ddot{r} \) can be moved outside sigma. What remains within sigma, according to \( \sum_{i=1}^{p} m^i s^i = 0 \), is equal to zero!

\[ \sum_{i=1}^{p} (m^i \ddot{s}^i \dddot{r}) = \sum_{i=1}^{p} (m^i \dddot{s}^i) \dddot{r} = 0 \]

(b) From Lesson 10 we have \( \ddot{s} = \dot{\omega} s \), \( \dddot{s} = -\ddot{s} \dot{\omega} + \dddot{\omega} \ddot{s} \) where \( \omega \) is the angular velocity of the body. Substituting in (b) and referring to Problem 2.15 yields

\[ \sum_{i=1}^{p} m^i \dot{s}^i \dddot{s}^i \Rightarrow \sum_{i=1}^{p} m^i \dot{s}^i (-\dddot{s}^i \dot{\omega} + \dddot{\omega} \ddot{s}^i) \Rightarrow \sum_{i=1}^{p} -m^i \ddot{s}^i \dot{s}^i \dot{\omega} + \sum_{i=1}^{p} m^i \dddot{s}^i \dddot{\omega} \ddot{s}^i \]

\[ \Rightarrow \sum_{i=1}^{p} -m^i \ddot{s}^i \dot{s}^i \dot{\omega} + \sum_{i=1}^{p} -m^i \dddot{s}^i \dddot{\omega} \ddot{s}^i \Rightarrow \sum_{i=1}^{p} -m^i \ddot{s}^i \dot{s}^i \dot{\omega} + \sum_{i=1}^{p} -m^i \dddot{\omega} \dddot{s}^i \ddot{s}^i \]

(c) This term represents the sum of all moments acting on the body

\[ n = \sum_{i=1}^{p} \dddot{s}^i f^i \]

(d) We can show that this term is exactly equal to zero! If we expand the terms within the sigmas, we can pair every two terms as shown:

\[ \sum_{i=1}^{p} \dddot{s}^i \sum_{j=1}^{p} f^{i,j} \Rightarrow \dddot{s}^i f^{i,i} + \dddot{s}^i f^{i,j} + \cdots \Rightarrow \dddot{s}^i f^{i,i} + \dddot{s}^i f^{i,j} + \cdots \Rightarrow (\dddot{s}^i - \dddot{s}^j) f^{i,j} + \cdots \]

According to the figure the vector \( s^{i,i} = s^i - s^j \) is parallel to the reaction force \( f^{i,j} \). Therefore, the product \( (\dddot{s}^i - \dddot{s}^j) f^{i,j} \) is zero!

Now the equation of motion has been simplified to

\[ \left( \sum_{i=1}^{p} -m^i \dddot{s}^i \dddot{s}^i \right) \dddot{\omega} + \dddot{\omega} \left( \sum_{i=1}^{p} -m^i \dddot{s}^i \dddot{s}^i \right) \dddot{\omega} = n \]

We replace each sigma by integral (infinite number of particles) and then introduce the inertia matrix as \( J \equiv \sum_{i=1}^{p} -m^i \dddot{s}^i s^i = \int_{\text{vol.}} -s \dddot{s} dm \)

The rotational equation of motion becomes

\[ J \dddot{\omega} + \dddot{\omega} J \dddot{\omega} = n \]

The inertia matrix \( J \) is obtained with respect to a reference frame attached to the mass center of the body and remaining parallel to the nonmoving \( x-y-z \) frame. Therefore
the components of \( \mathbf{J} \) vary with changing orientation of the body \( \mathbf{J}' \)

- The inertia matrix \( \mathbf{J}' \) is defined as

\[
\mathbf{J}' = \mathbf{A}^T \mathbf{J} \mathbf{A}
\]

The components of this matrix are constants—they are obtained with respect to a \( \xi - \eta - \zeta \) frame that is attached to and rotates with the body

- The rotational equation of motion can also be expressed as

\[
\mathbf{J}' \mathbf{\ddot{\omega}} + \mathbf{\dot{\omega}}' \mathbf{J}' \mathbf{\dot{\omega}} = \mathbf{n}'
\]

Equations of Motion for A Rigid Body

- **Newton-Euler equations**

  - Both the translational and rotational equations are described in the \( x-y-z \) components

\[
m \mathbf{\ddot{r}} = \mathbf{f} \\
\mathbf{J} \mathbf{\ddot{\omega}} + \mathbf{\dot{\omega}} \mathbf{J} \mathbf{\dot{\omega}} = \mathbf{n}
\]

  or,

\[
\begin{bmatrix}
m & 0 \\
0 & \mathbf{J}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{r}} \\
\mathbf{\ddot{\omega}}
\end{bmatrix}
+
\begin{bmatrix}
0 \\
\mathbf{\dot{\omega}} \mathbf{J} \mathbf{\dot{\omega}}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f} \\
\mathbf{n}
\end{bmatrix}
\]

  or,

\[
\begin{bmatrix}
m & 0 \\
0 & \mathbf{J}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{r}} \\
\mathbf{\ddot{\omega}}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{f} \\
\mathbf{n} - \mathbf{\dot{\omega}} \mathbf{J} \mathbf{\dot{\omega}}
\end{bmatrix}
\]

  (a)

  - The translational (Newton) equations are described in the \( x-y-z \) components but the rotational (Euler) equations are described in the \( \xi - \eta - \zeta \) components

\[
m \mathbf{\ddot{r}} = \mathbf{f} \\
\mathbf{J}' \mathbf{\ddot{\omega}} + \mathbf{\dot{\omega}}' \mathbf{J}' \mathbf{\dot{\omega}} = \mathbf{n}'
\]

  or,

\[
\begin{bmatrix}
m & 0 \\
0 & \mathbf{J}'
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{r}} \\
\mathbf{\ddot{\omega}}
\end{bmatrix}
+
\begin{bmatrix}
0 \\
\mathbf{\dot{\omega}}' \mathbf{J}' \mathbf{\dot{\omega}}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f} \\
\mathbf{n}'
\end{bmatrix}
\]

  or,

\[
\begin{bmatrix}
m & 0 \\
0 & \mathbf{J}'
\end{bmatrix}
\begin{bmatrix}
\mathbf{\ddot{r}} \\
\mathbf{\ddot{\omega}}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{f} \\
\mathbf{n}' - \mathbf{\dot{\omega}}' \mathbf{J}' \mathbf{\dot{\omega}}
\end{bmatrix}
\]

  (b)

  - Refer to the textbook for further discussion on these equations!

- When Euler equations are described in terms of their local components; i.e., Eq. (b), the inertia matrix \( \mathbf{J}' \) remains a constant. This could be a big advantage over Eq. (a) where the inertia matrix \( \mathbf{J} \) needs to be re-evaluated every time the rotational orientation of the body changes!

- We can derive the multibody equations of motion based on Eq. (a) or Eq. (b). If we derive the equations in one form, it takes a simple transformation to obtain the equations in the other form!

- The rotational equations of motion can also be derived in terms of the second time derivative of Euler parameters (this is done in the textbook—we will not use this form in this course!)
- In this course, we derive and use the equations of motion in the form of (a). This will be consistent with the kinematic constraints from Lesson 11.

- These equations for body $i$ are expressed in compact form as

$$ M_i \ddot{v}_i = g_i $$

where,

$$ M_i = \begin{bmatrix} m_i & 0 \\ 0 & J_i \end{bmatrix}, \quad \ddot{v}_i = \begin{bmatrix} \dddot{r}_i \\ \dddot{\omega}_i \end{bmatrix}, \quad g_i = \begin{bmatrix} f_i \\ n_i - \dddot{\omega}_i J_i \dot{\omega}_i \end{bmatrix} $$

- Note that $M_i$ and $g_i$ in your textbook (Eqs. 8.36, 8.38, and 8.39) are defined differently!

**A System of Unconstrained Bodies**

- In an unconstrained multibody system there are no kinematic joints; hence there are no kinematic constraints.

- The springs, dampers, actuators are force elements—it is assumed that a force element does not impose any constraints on a system.

  - In the next lesson we will learn how to construct the array of forces for several commonly used force elements

- Equations of motion are derived by constructing the Newton-Euler equations for every body in the system. Assume that there are $b$ bodies in the system:

$$ \mathbf{M} \ddot{\mathbf{v}} = \mathbf{g} $$

where

$$ \mathbf{M} = \begin{bmatrix} M_1 & & \\ & M_2 & \\ & & \ddots \\ & & & M_b \end{bmatrix}, \quad \ddot{\mathbf{v}} = \begin{bmatrix} \dddot{r}_1 \\ \dddot{r}_2 \\ \vdots \\ \dddot{r}_b \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_b \end{bmatrix} $$

**A System of Constrained Bodies**

- In a constrained multibody system one or more kinematic joints are present; hence kinematic constraints must be incorporated into the equations of motion. For a system of $b$ bodies the equations of motion are expressed as

$$ \mathbf{M} \ddot{\mathbf{v}} = \mathbf{g} + \mathbf{g}^{(c)} $$

where $\mathbf{g}^{(c)}$ represents the array of reaction forces, and $\mathbf{g}$ contains the spring, damper, ... forces and moments. Other elements in the above equation are the same as in the unconstrained equations of motion.

  - How do we determine the reaction forces?

- Assume that the constraints and their first and second time derivatives are represented as
\[ \Phi = \Phi(q) = 0 \]
\[ \Phi = Dv = 0 \]
\[ \Phi = \dot{D}v + Dv = 0 \]

The Jacobian matrix, \( D \), is used in determining the array of reaction forces as
\[ g^{(c)} = D^T \lambda \]
where \( \lambda \) contains as many coefficients as the number of constraints.

- These coefficients are called Lagrange multipliers
- At this point these multipliers are unknowns! In the upcoming lessons we will learn how to determine these multipliers.

- The equations of motion can be written as
  \[ M \ddot{v} = D^T \lambda \]
- In Section 8.4.3 of the textbook you find further discussion on the array of reaction forces
- In the upcoming lessons we will look at the reaction forces and moments for some specific kinematic joints.