**Lesson 9: Euler Parameters**

**Euler Theorem**
- According to the Euler Theorem, there exist a unique axis that if the $x$-$y$-$z$ frame (or the $\xi$--$\eta$--$\zeta$ frame) is rotated about it by an angle $\phi$ it becomes parallel to the $\xi$--$\eta$--$\zeta$ frame (or the $x$-$y$-$z$ frame). This axis is denoted by $\vec{u}$ and it is called the orientational axis of rotation.

**Euler Parameters**
- A set of rotational coordinates known as Euler parameters are defined as
  \[
  e_0 = \cos \frac{\phi}{2} \\
  \mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \mathbf{u} \sin \frac{\phi}{2}
  \]
  where $\mathbf{e}$ contains the $x$-$y$-$z$ or $\xi$--$\eta$--$\zeta$ components of $\vec{e}$
  - Vector $\vec{e}$ is along the orientational axis of rotation having a magnitude of $\sin \frac{\phi}{2}$
- The Euler parameters are denoted in any of the following forms:
  \[
  \mathbf{p} = \begin{pmatrix} e_0 \\ \mathbf{e} \end{pmatrix} = \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}
  \]
- The four Euler parameters are not independent--they are related through one constraint equation that can be expressed in any of the following forms:
  \[
  e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \\
  e_0^2 + \mathbf{e}^\top \mathbf{e} = 1 \\
  \mathbf{p}^\top \mathbf{p} = 1
  \]
- This process does not tell us how to locate the orientational axis of rotation!
- The transformation matrix is expressed as:
  \[
  \mathbf{A} = 2 \begin{pmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\ e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\ -e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_1^2 - \frac{1}{2} \end{pmatrix}
  \]
  In compact form this matrix is written as
  \[
  \mathbf{A} = (2 e_0^2 - 1) \mathbf{I} + 2 (\mathbf{e} \mathbf{e}^\top + e_0 \vec{e})
  \]
  - Note that the elements of this matrix are quadratic in terms of the Euler parameters
Inverse Problem

Assume that the values of the nine direction cosines; i.e., all the nine elements of the transformation matrix, are known. How do we determine the four Euler parameters?

\[
\begin{bmatrix}
  e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\
  e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\
  e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

We first equate the traces of the two matrices:

\[trA = a_{11} + a_{22} + a_{33}\]

\[= 2\left((e_0^2 + e_1^2 - \frac{1}{2}) + (e_0^2 + e_2^2 - \frac{1}{2}) + (e_0^2 + e_3^2 - \frac{1}{2})\right)\]

This yields the value of \(e_0^2\):

\[e_0^2 = \frac{trA + 1}{4}\]  \(a\)

The other three parameters are found by equating the diagonal terms:

\[
\begin{bmatrix}
  e_0^2 + e_1^2 - \frac{1}{2} & e_1 e_2 - e_0 e_3 & e_1 e_3 + e_0 e_2 \\
  e_1 e_2 + e_0 e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2 e_3 - e_0 e_1 \\
  e_1 e_3 - e_0 e_2 & e_2 e_3 + e_0 e_1 & e_0^2 + e_3^2 - \frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

We find:

\[e_1^2 = \frac{1 + 2a_{11} - trA}{4}, \quad e_2^2 = \frac{1 + 2a_{22} - trA}{4}, \quad e_3^2 = \frac{1 + 2a_{33} - trA}{4}\]  \(b\)

We note that these formulas give the values of the squared Euler parameters—how do we determine the correct signs for the parameters?

In order to determine the correct signs for the Euler parameters, we first find the value of \(e_0^2\) from \(a\). Assume that \(e_0^2 \neq 0\). We assign either a positive or a negative sign to \(e_0\). We then subtract the off-diagonal terms to compute the other three parameters

\[e_1 = \frac{a_{32} - a_{23}}{4e_0}, \quad e_2 = \frac{a_{13} - a_{31}}{4e_0}, \quad e_3 = \frac{a_{21} - a_{12}}{4e_0}\]  \(c\)

Since \(e_0 \neq 0\), we can find the other three parameters without any difficulties.

Note: It makes no difference what sign we give to \(e_0\) (why?)

What if \(e_0 = 0\)? We add the off-diagonal terms of the transformation matrix to obtain the following equations:

\[a_{21} + a_{12} = 4e_1 e_2\]  \(d\)
\[a_{13} + a_{31} = 4e_1 e_3\]  \(e\)
\[a_{32} + a_{23} = 4e_2 e_3\]  \(f\)

Compute \(e_1^2, e_2^2,\) and \(e_3^2\) from \(b\). At least one of these parameters must be nonzero!
— If \( e_1 \neq 0 \), use (d) and (e) to compute \( e_2 \) and \( e_3 \). It makes no difference what sign we give to \( e_1 \) (why?)

— If \( e_2 \neq 0 \), use (d) and (f) to compute \( e_1 \) and \( e_3 \); etc.

**Singularity?**

- The inverse process in determining the Euler parameters shows that there is **no singularity** associated with these parameters—as long as the direction cosines are known, we can find the corresponding Euler parameters! This is a major advantage in using these parameters.

- Another advantage of these parameters over the Euler angles is the transformation matrix is simpler and more efficient to compute.

- The disadvantage is that we need to make sure that the constraint on Euler parameters is satisfied; i.e., the sum of square of the four parameters must be exactly equal to one!

**Determining Euler Parameters**

**From Direction Cosines**

- In the previous lesson we discussed the necessary formulas for determining the Euler parameters if the direction cosines (the transformation matrix) are known.

- The direction cosines can be determined if we have a set of three angles describing the rotation (Euler angles), or if we have the coordinates of several points that are attached to the body.

**From Euler Angles**

- Assume that the three Euler angles (z-x-z convention) are known. What are the corresponding Euler parameters?

- We equate the two transformation matrices

\[
\begin{bmatrix}
2e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\
2e_1e_2 + e_0e_3 & 2e_2^2 + e_0^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\
e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2}
\end{bmatrix} =
\begin{bmatrix}
c\psi c\sigma - s\psi c\theta s\sigma & -c\psi s\sigma - s\psi c\theta c\sigma & s\psi s\theta \\
 s\psi c\sigma + c\psi c\theta s\sigma & -s\psi s\sigma + c\psi c\theta c\sigma & -c\psi s\theta \\
 s\theta s\sigma & s\theta c\sigma & c\theta
\end{bmatrix}
\]

- We compute the trace of both matrices to obtain \( e_0 \), then we equate the diagonal terms to get

\[
e_0 = \cos \frac{\theta}{2} \cos \frac{\psi + \sigma}{2} \\
e_1 = \sin \frac{\theta}{2} \cos \frac{\psi - \sigma}{2} \\
e_2 = \sin \frac{\theta}{2} \sin \frac{\psi - \sigma}{2} \\
e_3 = \cos \frac{\theta}{2} \sin \frac{\psi + \sigma}{2}
\]

- It is obvious that if we have the three angles we can always compute the four parameters.

**Inverse Conversion**

- The four Euler parameters are known, what are the corresponding Euler angles?
The conversion formulas are

\[
\begin{align*}
\cos \theta &= 2(e_0^2 + e_1^2) - 1 \\
\cos \sigma &= \frac{2(e_0 e_1 + e_0 e_3)}{\sin \theta} \quad \sin \sigma = \frac{2(e_0 e_1 - e_0 e_3)}{\sin \theta} \\
\cos \psi &= \frac{2(e_1 e_2 + e_3)}{\sin \theta} \quad \sin \psi = \frac{2(e_1 e_2 - e_3)}{\sin \theta}
\end{align*}
\]

We note that the singularity problem still exists when \( \sin \theta = 0 \)

The singularity problem is not caused by the Euler parameters—it is associated with the Euler angles!

From Coordinates of Points

- Assume that the \( x-y-z \) coordinates of four points, \( O, A, B \) and \( C \) are known
  - We compute the components of three vectors
    \[
    a = r^A - r^O \quad b = r^B - r^O \quad c = r^C - r^O
    \]
  - The vectors are normalized to obtain three unit vectors
    \[
    u_{(\xi)} = \frac{1}{a} a \quad u_{(\eta)} = \frac{1}{b} b \quad u_{(\zeta)} = \frac{1}{c} c
    \]
  - Now we have the nine direction cosines:
    \[
    A = \begin{bmatrix} u_{(\xi)} & u_{(\eta)} & u_{(\zeta)} \end{bmatrix}
    \]

- This process can also be achieved with only three points, \( O, A, \) and \( B \) as shown
  - We compute the components of two vectors
    \[
    a = r^A - r^O \quad b = r^B - r^O
    \]
  - The vectors are normalized to obtain two unit vectors
    \[
    u_{(\xi)} = \frac{1}{a} a \quad u_{(\eta)} = \frac{1}{b} b
    \]
  - The third unit vector is obtained as
    \[
    u_{(\zeta)} = u_{(\xi)} u_{(\eta)}
    \]
  - The transformation matrix is
    \[
    A = \begin{bmatrix} u_{(\xi)} & u_{(\eta)} & u_{(\zeta)} \end{bmatrix}
    \]

Special cases

- The Euler parameters can be found easily in the following special cases
  - The \( x-y-z \) frame and the \( \xi-\eta-\zeta \) frame are parallel
    \[
    p = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T
    \]
    Any axis can be the orientational axis of rotation; the angle of rotation is zero
The $x$-axis is parallel to the $\xi$-axis
$$p = \left\{ \begin{array}{ccc} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} & 0 \\ \frac{\phi}{2} & 0 & 0 \end{array} \right\}^T$$

The $x$-axis (or $\xi$-axis) is the orientational axis of rotation

The $y$-axis is parallel to the $\eta$-axis
$$p = \left\{ \begin{array}{ccc} \cos \frac{\phi}{2} & 0 & \sin \frac{\phi}{2} \\ \frac{\phi}{2} & 0 & 0 \end{array} \right\}^T$$

The $y$-axis (or $\eta$-axis) is the orientational axis of rotation

The $z$-axis is parallel to the $\zeta$-axis
$$p = \left\{ \begin{array}{ccc} \cos \frac{\phi}{2} & 0 & \sin \frac{\phi}{2} \\ \frac{\phi}{2} & 0 & 0 \end{array} \right\}^T$$

The $z$-axis (or $\zeta$-axis) is the orientational axis of rotation

Matrices and Identities with Euler Parameters

**G and L Matrices**

- Two $3 \times 4$ matrices are defined as
  $$G = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \quad L = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & e_1 & e_0 \end{bmatrix}$$

  In compact form:
  $$G = \begin{bmatrix} -e & -e_1 & e_0 \end{bmatrix} \quad L = \begin{bmatrix} -e & -e_1 & e_0 \end{bmatrix}$$

- The transformation matrix $A$ can be expressed as
  $$A = GL^T$$

- Since $G$ and $L$ matrices are linear in Euler parameters, it is obvious now why $A$ is quadratic in Euler parameters!

  These two matrices have useful characteristics where some identities are listed here. Refer to your textbook for the proof of these identities

  $$Gp = 0 \quad GG^T = I \quad G^T G = -pp^T + I \quad Gp = -Gp \quad \dot{Gp} = 0$$

  $$Lp = 0 \quad LL^T = I \quad L^T L = -pp^T + I \quad Lp = -Lp \quad \dot{Lp} = 0$$

  where $I^T$ is a $4 \times 4$ identity matrix.
• Note that most of these identities are interchangeable between $G$ and $L$
• These and many other identities can become useful in the derivation of the kinematic constraints and the equations of motion

Constraints on Euler Parameters
• We have already seen that the four Euler parameters must satisfy the following constraint
  \[ p^T p - 1 = 0 \]
• The time derivative of this constraint provides the constraint on the first time derivative of Euler parameters
  \[ p^T \dot{p} = 0 \]
• The time derivative of the velocity constraint provides the constraint on the second time derivative of Euler parameters
  \[ p^T \ddot{p} + p^T \dot{p} = 0 \]

Additional Matrices and Identities
• Two $4 \times 4$ matrices in terms of the elements of a 3-vector $a$ are defined as
  \[ a^+ \equiv \begin{bmatrix} 0 & -a^T \\ a & \dot{a} \end{bmatrix} \quad a^- \equiv \begin{bmatrix} 0 & -a^T \\ a & -\dot{a} \end{bmatrix} \]
• These matrices can be used in deriving many identities to be used in the derivation of constraint equations, etc.
  \[ G^T a = a^p \quad G a = \dot{a} G + a p^T \quad \dot{G}^T a = a \dot{p} \]
  \[ L^T a = a^- \quad L a = -\dot{a} L + a p^T \quad \dot{L}^T a = \dot{a} \dot{p} \]
• Other useful identities:
  \[ \dot{A} a = 2 G a \dot{p} \quad \dot{A}^T a = 2 L a \dot{p} \]
  \[ \frac{\partial}{\partial p} (A a) = 2 G a + 2 a p^T \quad \frac{\partial}{\partial p} (A^T a) = 2 L a + 2 a p^T \]
• There is no need for us to memorize any of these identities. However, it will be useful to understand what they are and how to use them. Check the textbook and learn how some of these identities are derived. Some of these identities can be used to derive the partial derivative of a constraint equation with respect to Euler parameters. Other identities can be used to determine the first and second time derivatives of a constraint equation.