Lesson 8-B: Bryant Angles

x-y-z Convention of Euler Angles

• Bryant angles are the x-y-z convention of the Euler angles
• The x-y-z frame is rotated three times: first about the x-axis by an angle $\phi_1$; then about the new y-axis by an angle $\phi_2$; then about the newest z-axis by an angle $\phi_3$. If the three angles are chosen correctly, then the rotated frame will coincide with the $\xi-\eta-\zeta$ frame.

• The transformation matrix is found by considering three planar transformation matrices
\[
D = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi_1 & -\sin \phi_1 \\
0 & \sin \phi_1 & \cos \phi_1
\end{bmatrix},
C = \begin{bmatrix}
\cos \phi_2 & 0 & \sin \phi_2 \\
0 & 1 & 0 \\
-\sin \phi_2 & 0 & \cos \phi_2
\end{bmatrix},
B = \begin{bmatrix}
\cos \phi_3 & -\sin \phi_3 & 0 \\
\sin \phi_3 & \cos \phi_3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

• The transformation matrix $A$ is the product of these three planar transformation matrices
\[
A = DCB = \begin{bmatrix}
1 & 0 & 0 & \cos \phi_2 & 0 & \sin \phi_2 & \cos \phi_3 & -\sin \phi_3 & 0 \\
0 & \cos \phi_1 & -\sin \phi_1 & 0 & 1 & 0 & \sin \phi_3 & \cos \phi_3 & 0 \\
0 & \sin \phi_1 & \cos \phi_1 & -\sin \phi_2 & 0 & \cos \phi_2 & 0 & 0 & 1
\end{bmatrix}
\]

where: $c \equiv \cos$ and $s \equiv \sin$

• Note that the resulting transformation matrix, similar to the z-x-z convention, is highly nonlinear in terms of the three angles
• This process does not tell us how to chose the value for each angle! If the angles are not chosen correctly, following the rotations, the x-y-z frame will not coincide with the $\xi-\eta-\zeta$ frame!
• We have the same problem of “singularity” as in the z-x-z convention!

Inverse Problem

• Assume that the values of the nine direction cosines; i.e., all the nine elements of the transformation matrix, are known. How do we determine the three Bryant angles?
• We equate some of the direction cosines with the entries of the transformation matrix $A$: 
Then we can write

$$\sin \phi_2 = a_{13} \quad \sin \phi_1 = \frac{-a_{23}}{\cos \phi_2} \quad \sin \phi_3 = \frac{-a_{12}}{\cos \phi_2}$$

$$\cos \phi_1 = \frac{a_{33}}{\cos \phi_2} \quad \cos \phi_3 = \frac{a_{11}}{\cos \phi_2}$$

-- The formula $\sin \phi_2 = a_{13}$ provides two answers for $\phi_2$; use one of them! (Does it make any difference which one we choose?)

-- Formulas for $\sin \phi_1$ and $\cos \phi_1$ provide a single value for $\phi_1$

-- Formulas for $\sin \phi_3$ and $\cos \phi_3$ provide a single value for $\phi_3$

-- The three angles can be determined only if $\cos \phi_2 \neq 0$

Singularity

- When $\cos \phi_2 = 0$, we fail to compute the other two angles
- This occurs when $\phi_2 = \frac{\pi}{2}, \frac{3\pi}{2}, ...$
- The transformation matrix finds the following form when $\cos \phi_2 = 0$:

$$A = \begin{bmatrix}
0 & 0 & \pm1 \\
a_{21} & a_{22} & 0 \\
a_{31} & a_{32} & 0
\end{bmatrix}$$

- Graphically, the axes of the first and third rotations coincide; i.e., we have rotated about the original x-axis twice. Therefore, the first and the third angles can be combined as one rotation. (Show this graphically!)

- The singularity phenomenon is an inherent problem associated with any three rotational coordinates regardless of their forms or convention

- This problem can be avoided if we use four rotational coordinates as we will see in the next lesson