Lesson 8-A: Euler Angles

Reference Frames

• In order to concentrate on the rotational coordinates of a body, we eliminate the translational coordinates by allowing the two reference frames \( \xi-\eta-\zeta \) and \( x-y-z \) to coincide at the origins.

Planar Rotation in Space

• Three planar rotations:
  - Assume that we perform a planar rotation in space, e.g., we perform a planar rotation in the \( x-y \) plane (\( \xi-\eta \) plane) by rotating about the \( z \)-axis (or \( \zeta \)-axis). The transformation matrix for this rotation is
    \[
    \mathbf{A} = \begin{bmatrix}
    \cos \phi & -\sin \phi & 0 \\
    \sin \phi & \cos \phi & 0 \\
    0 & 0 & 1
    \end{bmatrix}
    \]
  - Rotation about \( x \)-axis (or \( \xi \)-axis)
    \[
    \mathbf{A} = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos \phi & -\sin \phi \\
    0 & \sin \phi & \cos \phi
    \end{bmatrix}
    \]
  - Rotation about \( y \)-axis (or \( \eta \)-axis)
    \[
    \mathbf{A} = \begin{bmatrix}
    \cos \phi & 0 & \sin \phi \\
    0 & 1 & 0 \\
    -\sin \phi & 0 & \cos \phi
    \end{bmatrix}
    \]

Note the signs for the “\( \sin \phi \)” terms!

Euler Angles

• Euler angles are the most commonly used rotational coordinates
• There are many different conventions of Euler angles
• All conventions are the result of three consecutive rotations about three different axes
• Depending on the choice of rotational axes, different conventions are found
The most common convention is the z-x-z convention (initially defined for gyroscopes).

Another common convention is the x-y-z also known as the Bryant angles.

There is an inherent problem associated with any of these conventions known as the singularity problem.

**z-x-z Convention of Euler Angles**

In the z-x-z convention, the x-y-z frame is rotated three times: first about the z-axis by an angle \( \psi \); then about the new x-axis by an angle \( \theta \); then about the newest z-axis by an angle \( \sigma \). If the three rotational angles are chosen correctly, the rotated frame will coincide with the \( \xi-\eta-\zeta \) frame.

The transformation matrix is found by considering three planar transformation matrices:

\[
D = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\cos \sigma & -\sin \sigma & 0 \\
\sin \sigma & \cos \sigma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The transformation matrix \( A \) is the product of these three planar transformation matrices:

\[
A = DCB = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \sigma & -\sin \sigma & 0 \\
\sin \sigma & \cos \sigma & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
c\psi c\sigma - s\psi c\theta s\sigma & -c\psi s\sigma - s\psi c\theta c\sigma & s\psi s\theta \\
s\psi c\sigma + c\psi c\theta s\sigma & -s\psi s\sigma + c\psi c\theta c\sigma & -c\psi s\theta \\
s\theta s\sigma & s\theta c\sigma & c\theta
\end{bmatrix}
\]

where: \( c \equiv \cos \) and \( s \equiv \sin \)

Note that the resulting transformation matrix is highly nonlinear in terms of the three angles.

This process does not tell us how to choose the value for each angle! If the angles are not chosen correctly, following the rotations, the x-y-z frame will not coincide with the \( \xi-\eta-\zeta \) frame!

Furthermore, there is a problem of “singularity” that we need to be aware of!

**Inverse Problem**

Assume that the values of the nine direction cosines; i.e., all the nine elements of the transformation matrix, are known. How do we determine the three Euler angles?
We equate some of the direction cosines with the entries of the transformation matrix $A$:

$$
\begin{bmatrix}
c\psi c\sigma - s\psi c\theta s\sigma & -c\psi s\sigma - s\psi c\theta c\sigma & s\psi s\theta \\
- s\psi c\sigma + c\psi c\theta s\sigma & c\psi s\sigma + s\psi c\theta c\sigma & -c\psi s\theta \\
s\theta s\sigma & s\theta c\sigma & c\theta
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
$$

Then we can write

$$
\cos \theta = a_{13} \quad \cos \psi = \frac{-a_{23}}{\sin \theta} \quad \cos \sigma = \frac{a_{32}}{\sin \theta} \\
\sin \psi = \frac{a_{13}}{\sin \theta} \quad \sin \sigma = \frac{a_{31}}{\sin \theta}
$$

-- The formula $\cos \theta = a_{13}$ provides two answers for $\theta$; use one of them! (Does it make any
difference which one we choose?)
-- Formulas for $\cos \psi$ and $\sin \psi$ provide a single value for $\psi$
-- Formulas for $\cos \sigma$ and $\sin \sigma$ provide a single value for $\sigma$
-- The three angles can be determined only if $\sin \theta \neq 0$

**Singularity**

- When $\sin \theta = 0$ we fail to compute the other two Euler angles
- This occurs when $\theta = 0, \pi, 2\pi, ...$
- The transformation matrix finds the following form when $\sin \theta = 0$:

$$
A = \begin{bmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & 0 \\
0 & 0 & \pm 1
\end{bmatrix}
$$

- Graphically, the axes of the first and third rotations coincide; i.e., we have rotated about the
original $z$-axis twice. Therefore, the first and the third angles can be combined as one
rotation!