Lesson 7: Kinematics of A Spatial Rigid Body

Reference Frames

- In a multibody system, we need to define:
  - One non-moving (also called global, absolute, or inertial) frame: \( x-y-z \)
  - One body-fixed frame (also called local) per body: \( \xi-\eta-\zeta \)
- A body-fixed frame carries the corresponding body index (subscript)

Array (Vector) of Coordinates

- Array of coordinates for a rigid body can be defined as
  \[
  \mathbf{q}_i = \begin{cases} \mathbf{r}_i \end{cases} \text{ or } \mathbf{q}_i = \begin{cases} \mathbf{p}_i \end{cases}
  \]
  where \( \mathbf{r}_i = \{x_i, y_i, z_i\}^T \) are the Cartesian coordinates of the origin of the body-fixed frame
- Rotational coordinates could be represented by three angles as \( \phi = \{\phi_i, \theta_i, \sigma_i\}^T \) or by four parameters as \( \mathbf{p}_i = \{e_0, e_1, e_2, e_3\}^T \)

- In the following lessons we will discuss how to describe rotational coordinates for a rigid body. Two forms of rotational coordinates will specifically be discussed:
  - Euler angles (3 coordinates)
  - Euler parameters (4 coordinates)
- We will see how these rotational coordinates can be used to construct a rotational transformation matrix for a body

Components of A Vector

- Vector \( \mathbf{s} \) is attached to a body
  - Components in \( x-y-z \) and in \( \xi-\eta-\zeta \)
    \[
    \mathbf{s}_{(x-y-z)} \equiv \begin{bmatrix} s_{(x)} \\ s_{(y)} \\ s_{(z)} \end{bmatrix}, \quad \mathbf{s}_{(\xi-\eta-\zeta)} \equiv \begin{bmatrix} s_{(\xi)} \\ s_{(\eta)} \\ s_{(\zeta)} \end{bmatrix}
    \]
  - Transformation: \( \mathbf{s} = \mathbf{A} \mathbf{s}' \)

Transformation Matrix

- Matrix \( \mathbf{A} \) in \( \mathbf{s} = \mathbf{A} \mathbf{s}' \) is called rotational transformation matrix
- Matrix \( \mathbf{A} \) is a \( 3 \times 3 \) matrix; it transforms components of a vector from \( \xi-\eta-\zeta \) to \( x-y-z \) frame
The inverse transformation is \( s' = A^T s \).

\( A \) is an orthonormal matrix; i.e., \( A^{-1} = A^T \).

In some textbooks the rotational transformation matrix is defined as \( R \) such that \( s' = R s \); i.e., \( R = A^T \).

Elements of \( A \) are called direction cosines:

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

Columns of \( A \) contain components of three unit vectors \( \tilde{u}(\xi), \tilde{u}(\eta) \) and \( \tilde{u}(\zeta) \) projected onto the \( x-y-z \) axes:

\[
\begin{align*}
\tilde{u}(\xi) &= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, & \tilde{u}(\eta) &= \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, & \tilde{u}(\zeta) &= \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}
\end{align*}
\]

Note:

\[
\begin{align*}
\tilde{u}'(\xi) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \tilde{u}'(\eta) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \tilde{u}'(\zeta) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

Columns of \( A^T \) contain components of three unit vectors \( \tilde{u}(x), \tilde{u}(y) \) and \( \tilde{u}(z) \) projected onto the \( \xi-\eta-\zeta \) axes:

\[
\begin{align*}
\tilde{u}(x) &= \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix}, & \tilde{u}(y) &= \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix}, & \tilde{u}(z) &= \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}
\end{align*}
\]

Note:

\[
\begin{align*}
\tilde{u}'(x) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & \tilde{u}'(y) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & \tilde{u}'(z) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

**A Point on A Body**

Vector \( \tilde{s}^p \) locates point \( P \) with respect to the origin of \( \xi-\eta-\zeta \) frame.

- Local coordinates of \( P \):
  \[
  s_{\xi-\eta-\zeta}^p \equiv \begin{bmatrix} s_{\xi}^p \\ s_{\eta}^p \\ s_{\zeta}^p \end{bmatrix}
  \]

- Global coordinates of \( P \):
  \[
  r^p = r + s^p = r + A \cdot s^p
  \]