Lesson 2

Scalars

• A scalar is a quantity possessing only a magnitude
  — Examples
    • Mass, \( m \)
    • Length,
    • Volume, \( V \)
    • Speed, \( v \)
  — There is no direction associated with a scalar
  — Scalars are designated by light-face characters

Vector Representation

• A vector is a quantity possessing a magnitude and a direction
• In this course we use two types of vector representations:
  — Geometric
  — Algebraic

Geometric Vectors

• A geometric vector is represented by a lower-case, light-face, italic character with an over-score
• A typical vector \( \vec{s} \):
  — Magnitude of \( \vec{s} \) is shown as \(|s|\)
  — Examples:
    • Position vector: \( \vec{r} \)
    • Velocity vector: \( \vec{v} \)
    • Acceleration vector: \( \vec{a} \)
    • Force vector: \( \vec{f} \)

Unit Vector

• A unit vector has a magnitude of “1” unit
  — Example: Unit vector \( \vec{u} \)
A unit vector along the axis of vector \( \vec{a} \) may be shown as \( \hat{u}_{(a)} \).

Any vector can be described as the product of its magnitude and a unit vector defined along its axis: \( \vec{a} = a \hat{u}_{(a)} \).

**Scalar Product**
- The scalar (or dot) product of two vectors is defined as
  \[ \vec{a} \cdot \vec{b} = ab \cos \theta \]
- For any two vectors \( \vec{a} \) and \( \vec{b} \):
  \[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \]
- For any vector \( \vec{a} \):
  \[ \vec{a} \cdot \vec{a} = a^2 \]

**Orthogonal vectors:**
- Two vectors \( \vec{a} \) and \( \vec{b} \) are said to be orthogonal (normal or perpendicular) if:
  \[ \vec{a} \cdot \vec{b} = 0 \]

**Vector Product**
- Vector (or cross) product of two vectors \( \vec{a} \) and \( \vec{b} \) is defined as
  \[ \vec{c} = \vec{a} \times \vec{b} \]
  - Vector \( \vec{c} \) is perpendicular to the plane of \( \vec{a} \) and \( \vec{b} \)
  - The magnitude of \( \vec{c} \) is computed as \( c = ab \sin \theta \)
- For any two vectors \( \vec{a} \) and \( \vec{b} \):
  \[ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \]

**Parallel Vectors**
- In the diagram vectors \( \vec{a} \), \( \vec{b} \) and \( \vec{c} \) are parallel.
Collinear Vectors

- In the diagram vectors \( \vec{d} \) and \( \vec{e} \) are collinear.

Algebraic Vectors

- A 3D vector \( \vec{a} \) in an orthogonal coordinate frame is denoted in algebraic form as a column array (vector):

\[
\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}
\]

  - An algebraic row vector is written as \( \vec{a}^\text{r} = \{a_1, a_2, a_3\} \)
  - An algebraic vector is hand-written as \( \vec{a} \) (instead of \( a \))
  - Note that in the textbook square brackets, [ ], are used instead of curly, { }, brackets!

Algebraic Vector Operations

- The figure shows geometric vector summation and subtraction.
  - Algebraic vector summation:

\[
\vec{c} = \vec{a} + \vec{b}
\]

\[
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}
\]

  - Algebraic vector subtraction:

\[
\vec{c} = \vec{a} - \vec{b}
\]

\[
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}
\]

- Scalar product: \( \vec{a}^\text{T} \vec{b} = \{a_1, a_2, a_3\} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1b_1 + a_2b_2 + a_3b_3 

  - Note: \( \vec{b}^\text{T} \vec{a} = \vec{a}^\text{T} \vec{b} \)
- Skew-symmetric matrix:
For \( \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \) we define \( \hat{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \)

- Vector product: \( \hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{b}} \Rightarrow \mathbf{c} = \hat{\mathbf{a}} \hat{\mathbf{b}} \)
  - Note: \( \hat{\mathbf{a}} \hat{\mathbf{b}} = -\hat{\mathbf{b}} \hat{\mathbf{a}} \) and \( \hat{\mathbf{a}} \hat{\mathbf{a}} = 0 \)

- Useful identities (learn how to use them!):
  \[
  \hat{\mathbf{a}} \hat{\mathbf{b}} = \mathbf{b} \mathbf{a}^T - \mathbf{a}^T \mathbf{b} \mathbf{1} \\
  \sim \\
  \hat{\mathbf{a}} \hat{\mathbf{b}} = \mathbf{b} \mathbf{a}^T - \mathbf{a} \mathbf{b}^T \\
  = \hat{\mathbf{a}} \hat{\mathbf{b}} - \hat{\mathbf{b}} \hat{\mathbf{a}} \\
  \hat{\mathbf{a}} \hat{\mathbf{b}} + \mathbf{a} \mathbf{b}^T = \hat{\mathbf{b}} \hat{\mathbf{a}} + \mathbf{b} \mathbf{a}^T
  \]

  These identities can be verified by direct calculation (expanding them into components)

Arrays

- An array is a column vector of any dimension
  - A 6-array (example):
    \[
    \mathbf{a} = \begin{bmatrix} a_{(x)} \\ a_{(y)} \\ a_{(z)} \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}
    \]

- Different ways of representing an array:
  \[
  \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}^T
  \]

- An array may contain other arrays such as \( \mathbf{a} \), \( \mathbf{b} \), ..., \( \mathbf{d} \) with equal or different dimensions:
  \[
  \mathbf{r} = \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \\ \vdots \\ \mathbf{d} \end{bmatrix}
  \]
Matrices

- A typical $m \times n$ matrix:

$$
A = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
$$

- Number of rows: $m$
- Number of columns: $n$

- Square matrix: $m = n$

- Transpose

  Example: $B = \begin{bmatrix} 2 & a \\ -1 & 3 \end{bmatrix}$, $B^T = \begin{bmatrix} 2 & -1 \\ a & 3 \end{bmatrix}$

- Symmetric matrix

  $A = A^T$

- Diagonal matrix

  $D = \begin{bmatrix}
  d_{11} & 0 & \cdots & 0 \\
  0 & d_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_{nn}
\end{bmatrix}$

- Identity matrix

  $I = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{bmatrix}$

Matrix Operations

- Summation and Subtraction

  $C = A + B$ \hspace{1cm} $D = A - B$

- Multiplication by a scalar

  $E = \alpha A$

- Multiplication of two matrices

  $F = AB$

  Number of columns of $A$ must be equal to the number of rows of $B$

- Inverse of a matrix

  $A^{-1}A = I$
Inverse of $A$ is denoted as $A^{-1}$

Matrix must be (i) square and (ii) non-singular

If the inverse does not exist, the matrix is singular

A square singular matrix has redundant rows or columns

**Time Derivatives**

- **Scalars**
  \[
  \frac{d}{dt} x \implies \dot{x} \quad \frac{d}{dt} \left( \frac{d}{dt} x \right) \implies \ddot{x}
  \]

- **Vectors and Arrays**
  \[
  \frac{d}{dt} \mathbf{a} = \begin{bmatrix}
  \frac{d}{dt} a_1(t) \\
  \vdots \\
  \frac{d}{dt} a_n(t)
  \end{bmatrix} \equiv \dot{\mathbf{a}}
  \]

- **Matrices**
  \[
  \frac{d}{dt} \mathbf{B} = \begin{bmatrix}
  \frac{d}{dt} b_{11} & \cdots & \frac{d}{dt} b_{1n} \\
  \vdots & \ddots & \vdots \\
  \frac{d}{dt} b_{m1} & \cdots & \frac{d}{dt} b_{mn}
  \end{bmatrix} = \dot{\mathbf{B}}
  \]

- **Chain rule of differentiation:**
  \[
  \frac{d}{dt} (\alpha \mathbf{b}) = \alpha \dot{\mathbf{b}} + \dot{\alpha} \mathbf{b}
  \]
  \[
  \frac{d}{dt} (\mathbf{a}^T \mathbf{b}) = \mathbf{a}^T \dot{\mathbf{b}} + \mathbf{b}^T \dot{\mathbf{a}}
  \]
  \[
  \frac{d}{dt} (\mathbf{a} \mathbf{b}) = \dot{\mathbf{a}} \mathbf{b} + \mathbf{a} \dot{\mathbf{b}}
  \]
  \[
  \frac{d}{dt} (\mathbf{C} \mathbf{D}) = \dot{\mathbf{C}} \mathbf{D} + \mathbf{C} \dot{\mathbf{D}}
  \]

**Partial Derivatives**

- Assume that $\mathbf{q}$ is an $n$-array of variables
- Assume that $\Phi$ is a function of $\mathbf{q}$
- Partial derivative of $\Phi$ (a single function) with respect to $\mathbf{q}$ is a row matrix with $n$ columns ($1 \times n$):
  \[
  \frac{\partial \Phi}{\partial \mathbf{q}} = \begin{bmatrix}
  \frac{\partial \Phi}{\partial q_1} & \frac{\partial \Phi}{\partial q_2} & \cdots & \frac{\partial \Phi}{\partial q_n}
  \end{bmatrix}
  \]
- Assume that $\Phi$ is an $m$-array of differentiable functions of $\mathbf{q}$
Partial derivative of $\Phi$ with respect to $q$ is defined as an $m \times n$ matrix:

$$\frac{\partial \Phi}{\partial q} = \begin{bmatrix}
\frac{\partial \Phi_1}{\partial q_1} & \frac{\partial \Phi_1}{\partial q_n} & \cdots & \frac{\partial \Phi_1}{\partial q_n} \\
\frac{\partial \Phi_2}{\partial q_1} & \frac{\partial \Phi_2}{\partial q_n} & \cdots & \frac{\partial \Phi_2}{\partial q_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \Phi_m}{\partial q_1} & \frac{\partial \Phi_m}{\partial q_n} & \cdots & \frac{\partial \Phi_m}{\partial q_n}
\end{bmatrix}$$

Chain rule of differentiation must be applied to partial derivatives with care!

Assume that $a$ and $b$ are $n$-arrays functions of $q$

The partial derivative of the scalar product of $a$ and $b$ with respect to $q$ is found as:

$$\frac{\partial}{\partial q} (a^\top b) = b^\top \frac{\partial a}{\partial q} + a^\top \frac{\partial b}{\partial q}$$

The partial derivative of the vector product of $a$ and $b$ with respect to $q$ is found as:

$$\frac{\partial}{\partial q} (\hat{a} \times \hat{b}) = -\hat{b} \frac{\partial \hat{a}}{\partial q} + \hat{a} \frac{\partial \hat{b}}{\partial q}$$

Equations and Expressions

The term *Equation* is referred to a set of identities that needs to be solved simultaneously to find the unknowns

The term *Expression* is referred to one identity or a set of identities that requires only simple substitutions in order to find the unknowns

Unit System

In this course the choice of the unit system is left to the student

In most examples the unit system is not specified

In some problems and in the computer programs where the unit system needs to be specified, the SI units are used