Cauchy Sequences and Complete Metric Spaces

**Definition:** A sequence \( \{x_n\} \) in a metric space \((X, d)\) is **Cauchy** if
\[
\forall \epsilon > 0 : \exists n \in \mathbb{N} : m, n > n \Rightarrow d(x_m, x_n) < \epsilon.
\]

**Remark:** Every convergent sequence is Cauchy.

**Proof:** Let \( \{x_n\} \to \bar{x} \), let \( \epsilon > 0 \), let \( n \) be such that \( n > n \Rightarrow d(x_n, \bar{x}) < \epsilon/2 \), and let \( m, n > n \). Then
\[
d(x_m, \bar{x}) < \frac{\epsilon}{2} \quad \text{and} \quad d(x_n, \bar{x}) < \frac{\epsilon}{2},
\]
and the Triangle Inequality yields
\[
d(x_m, x_n) \leq d(x_m, \bar{x}) + d(x_n, \bar{x}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square
\]

**Exercise:** The real sequence \( \{x_n\} \) defined by \( x_n = \frac{1}{n} \) converges, so it is Cauchy. Prove directly that it is Cauchy, by showing how the \( n \) depends upon \( \epsilon \).

**Example 1:** Let \( x_n = \frac{1}{n} \sqrt{2} \) for each \( n \in \mathbb{N} \). Note that each \( x_n \) is an irrational number (\( i.e., \ x_n \in \mathbb{Q}^c \)) and that \( \{x_n\} \) converges to 0. Thus, \( \{x_n\} \) converges in \( \mathbb{R} \) (\( i.e., \), to an element of \( \mathbb{R} \)). But 0 is a rational number (thus, \( 0 \not\in \mathbb{Q}^c \)), so although the sequence \( \{x_n\} \) is entirely in \( \mathbb{Q}^c \), it does not converge in \( \mathbb{Q}^c \). Note, however, that \( \{x_n\} \) is Cauchy.

**Example 2:** Let \( \bar{x} \) be an irrational number, and for each \( n \in \mathbb{N} \) let \( \{x_n\} \) be a rational number in the interval \((\bar{x} - \frac{1}{n}, \bar{x} + \frac{1}{n})\). Then \( \{x_n\} \) is a sequence of rational numbers that converges to the irrational number \( \bar{x} \) — \( i.e., \), each \( x_n \) is in \( \mathbb{Q} \) and \( \{x_n\} \to \bar{x} \not\in \mathbb{Q} \). Thus, in a parallel to Example 1, \( \{x_n\} \) here is a Cauchy sequence in \( \mathbb{Q} \) that does not converge in \( \mathbb{Q} \).

Examples 1 and 2 demonstrate that both the irrational numbers, \( \mathbb{Q}^c \), and the rational numbers, \( \mathbb{Q} \), are not entirely well-behaved metric spaces — they are not complete in that there are Cauchy sequences in each space that don’t converge to an element of the space.

**Definition:** A metric space \((X, d)\) is **complete** if every Cauchy sequence in \( X \) converges in \( X \) (\( i.e., \), to a limit that’s in \( X \)).

**Example 3:** The real interval \((0, 1)\) with the usual metric is not a complete space: the sequence \( x_n = \frac{1}{n} \) is Cauchy but does not converge to an element of \((0, 1)\).

**Example 4:** The space \( \mathbb{R}^n \) with the usual (Euclidean) metric is complete. We haven’t shown this before, but we’ll do so momentarily.
Remark 1: Every Cauchy sequence in a metric space is bounded.

Proof: Exercise.

Remark 2: If a Cauchy sequence has a subsequence that converges to \( \overline{x} \), then the sequence converges to \( \overline{x} \).

Proof: Exercise.

In proving that \( \mathbb{R} \) is a complete metric space, we’ll make use of the following result:

**Proposition:** Every sequence of real numbers has a monotone subsequence.

Proof: Suppose the sequence \( \{x_n\} \) has no monotone increasing subsequence; we show that then it must have a monotone decreasing subsequence. The sequence \( \{x_n\} \) must have a first term, say \( x_{n_1} \), such that all subsequent terms are smaller (i.e., \( n > n_1 \Rightarrow x_n < x_{n_1} \)); otherwise \( \{x_n\} \) would have a monotone increasing subsequence. Similarly, the subsequence \( \{x_{n_1+1}, x_{n_1+2}, \ldots\} \) must have a first term \( x_{n_2} \) such that all subsequent terms are smaller; note that \( x_{n_1} > x_{n_2} \). Continuing for \( n_1, n_2, n_3, \ldots \), we have a subsequence \( \{x_{n_k}\} \) such that \( x_{n_1} > x_{n_2} > x_{n_3} > \ldots \), a monotone decreasing subsequence. \( \square \)

Now we’ll prove that \( \mathbb{R} \) is a complete metric space, and then use that fact to prove that the Euclidean space \( \mathbb{R}^n \) is complete.

**Theorem:** \( \mathbb{R} \) is a complete metric space — i.e., every Cauchy sequence of real numbers converges.

Proof: Let \( \{x_n\} \) be a Cauchy sequence. Remark 1 ensures that the sequence is bounded, and therefore that every subsequence is bounded. The proposition we just proved ensures that the sequence has a monotone subsequence. The Monotone Convergence Theorem ensures that this subsequence converges. And therefore Remark 2 ensures that the original sequence converges. \( \square \)

This proof used the Completeness Axiom of the real numbers — that \( \mathbb{R} \) has the LUB Property — via the Monotone Convergence Theorem. We could have gone instead in the other direction: taking “every Cauchy sequence of real numbers converges” to be the Completeness Axiom, and then proving that \( \mathbb{R} \) has the LUB Property.

**Theorem:** The normed vector space \( \mathbb{R}^n \) is a complete metric space.

Proof: Exercise.

**Example 5:** The closed unit interval \([0, 1]\) is a complete metric space (under the absolute-value metric). This is easy to prove, using the fact that \( \mathbb{R} \) is complete.
Example 6: The space $C[0,1]$ is complete. (We haven’t shown this yet.)

Exercise: In a previous exercise set we worked with a sequence of distribution functions $F_n$ defined by

$$F_n(x) = \begin{cases} nx, & \text{if } x \leq \frac{1}{n} \\ 1, & \text{if } x \geq \frac{1}{n}. \end{cases}$$

on the unit interval $[0,1]$ in $\mathbb{R}$. We showed that $\{F_n\}$ does not converge in $C[0,1]$. Therefore, if $\{F_n\}$ were Cauchy, $C[0,1]$ would not be complete. Verify that $\{F_n\}$ is not Cauchy.

Example 7: (Obtaining $\mathbb{R}$ as the completion of $\mathbb{Q}$.)

Let $S$ be the set of Cauchy sequences in $\mathbb{Q}$ — i.e., the set of Cauchy sequences of rational numbers — with the usual metric. Define a relation $\sim$ on $S$ as follows:

$$\{x_n\} \sim \{x'_n\} \text{ if } \forall \epsilon > 0 : \exists \bar{n} \in \mathbb{N} : m, n > \bar{n} \Rightarrow d(x_m, x'_n) < \epsilon$$

Let $\mathbb{Q}^* = S/\sim$, the partition of $S$ consisting of equivalence classes of Cauchy sequences. Define the distance function $d^*$ for $\mathbb{Q}^*$ as follows:

For any $x, x' \in \mathbb{Q}^*$, let $\{x_n\} \in x$ and $\{x'_n\} \in x'$ (i.e., $x = [\{x_n\}]$ and $x' = [\{x'_n\}]$).

Then define $d^*(x, x')$ by $d^*(x, x') = \lim_{n \to \infty} d(x_n, x'_n)$.

It’s pretty straightforward to show that $d^*$ is well-defined and is a metric for $\mathbb{Q}^*$. The metric space $(\mathbb{Q}^*, d^*)$ can be placed into one-to-one correspondence with $(\mathbb{R}, |\cdot|)$, each constant sequence $\{r, r, r, \ldots\}$ of rationals corresponding to the rational number $r \in \mathbb{Q} \subseteq \mathbb{R}$. The set $\mathbb{Q}^*$ is one way of defining $\mathbb{R}$.

Exercise: Verify that the relation $\sim$ defined in Example 7 is an equivalence relation.

Example 8: (The Completion of a Metric Space)

Let $(X, d)$ be a metric space that is not complete. Just as in Example 7, let $S$ be the set of Cauchy sequences in $X$; define the equivalence relation $\sim$ in the same way, and let $X^*$ be the quotient space $S/\sim$; and define $d^*$ on the quotient space $X^*$ in the same way as in Example 7. Then we can show, just as in Example 7, that $d^*$ is well-defined and is a metric for $X^*$; that $(X^*, d^*)$ is a complete metric space; and that $X$ corresponds to a subset of $X^*$ — we say that $X$ is embedded in $X^*$. The complete metric space $(X^*, d^*)$ is called the completion of $(X, d)$.

Example 9: The open unit interval $(0,1)$ in $\mathbb{R}$, with the usual metric, is an incomplete metric space. What is its completion, $((0,1)^*, d^*))$?

Theorem: A subset of a complete metric space is itself a complete metric space if and only if it is closed.

Proof: Exercise.
Recall that every normed vector space is a metric space, with the metric \( d(x, x') = \|x - x'\| \). Therefore our definition of a complete metric space applies to normed vector spaces: an n.v.s. is complete if it’s complete as a metric space, \( i.e.\), if all Cauchy sequences converge to elements of the n.v.s.

**Definition:** A complete normed vector space is called a **Banach space**.

**Example 4 revisited:** \( \mathbb{R}^n \) with the Euclidean norm is a Banach space.

**Example 6 revisited:** \( C[0, 1] \) is a Banach space.

**Example 5 revisited:** The unit interval \([0, 1]\) is a complete metric space, but it’s not a Banach space because it’s not a vector space.