This theorem tells us how a decision-maker's choice (behavior) respond to changes in the decision-making environment.

**Theorem:** Let $X \subseteq \mathbb{R}^n$ be the actions, $E \subseteq \mathbb{R}^m$ be the environments/parameters, $u : X \times E \rightarrow \mathbb{R}$ be the objective function, and $\phi : E \rightarrow X$ be the feasible set correspondence. We define

$$\mu : E \rightarrow X \text{ by } \mu(e) = \{ x \in \phi(e) \mid \text{max } u(x, e) \text{ in } \phi(e) \}$$

$$= \arg \max_{x \in \phi(e)} u(x, e) \quad \text{Behavioral (solution) correspondence}$$

and $v : E \rightarrow \mathbb{R} \text{ by } v(e) = \max_{x \in \phi(e)} u(x, e). \quad \text{Value function}$

If $u$ is continuous, and $\phi$ is continuous and compact-valued, then

$\mu$ is UHC and nonempty-valued, and $v$ is continuous.

Since $\mu$ is closed-valued and UHC,

**Remark:** $\mu$ has a closed graph.

**Remark:** If $\mu$ is singleton valued (a function), it is continuous.
The Maximum Theorem in Demand Theory

**Maximum Theorem**

\[ x \in X \subseteq \mathbb{R}^k \]

\[ e \in E \subseteq \mathbb{R}^m \]

\[ u : X \times E \rightarrow \mathbb{R} \]

\[ \varphi : E \rightarrow X \]

**Demand Theory**

\[ x \in \mathbb{R}^k_+, \text{ consumption bundles} \]

\[ p \in \mathbb{R}^k_+, \text{ price lists} \]

\[ u : \mathbb{R}^k_+ \times \mathbb{R}^m_+ \rightarrow \mathbb{R}, \text{ utility function} \]

\[ \beta(p, m) = \{ x \in \mathbb{R}^k_+ | p \cdot x \leq m \}, \text{ m must be p \cdot x} \]

Consumer's budget set

\[ \mu : E \rightarrow X \]

\[ \mu : \mathbb{R}^k_+ \rightarrow \mathbb{R}^k_+, \text{ consumer's demand correspondence} \]

\[ v : E \rightarrow \mathbb{R} \]

\[ v : \mathbb{R}^k_+ \rightarrow \mathbb{R}, \text{ indirect utility function} \]

The Maximum Theorem tells us that the consumer's demand correspondence \( \mu : \mathbb{R}^k_+ \rightarrow \mathbb{R}^k_+ \) is nonempty-valued, compact-valued, closed, and UHC if the budget correspondence is continuous and compact-valued. Since the budget correspondence is typically not compact-valued for price lists that have some \( p_k = 0 \), we need to either restrict prices to be in a subset \( \mathfrak{S} \) of \( \mathbb{R}^k_+ \), or else restrict the consumption set \( X \) to a compact subset of \( \mathbb{R}^k_+ \).

We also obtain:

The indirect utility function is continuous, and

If demand \( \mu(p) \) is singleton-valued then

The demand function is continuous.
Theory of the Firm:

A firm's technology (its technologically feasible production plans) is represented by its production set $T \subseteq \mathbb{R}^d$:

- $x_k > 0$: good $k$ is an output at $x$
- $x_k < 0$: good $k$ is an input at $x$

$T$ is the set of all technologically feasible production plans.

Profit at plan $x$ (at price list $p \in \mathbb{R}^d^+$):

$$\pi(x; p) = p \cdot x = \sum_{k \in T} p \cdot x_k + \sum_{k \in I} p \cdot x_k$$

$T = \{k \mid x_k > 0\}$, $I = \{k \mid x_k \leq 0\}$

$$\pi(x; p) = \text{REVENUE} - \text{COST}.$$

Constant Returns to Scale:

- $x_2 = y$
- $x_2 = f(-x_1)$
- $y = f(x)$

$T = \{x \mid \text{PRODUCTION FUNCTION}\}$

$-x_1 = x_2$

$-x_1 = -x_2$

$\pi = 0$, multiple max. points
The Firm's Behavioral Correspondence

\textbf{Maximum Theorem}

\begin{align*}
\text{Max} & \quad x \in \mathbb{R}^n \quad \text{subject to} \quad e \in \mathbb{R}^m \\
\text{Min} & \quad x \in \mathbb{R}^n \quad \text{subject to} \quad e \in \mathbb{R}^m \\
\text{Max} & \quad x \in \mathbb{R}^n \\
\text{Min} & \quad x \in \mathbb{R}^n \\
\end{align*}

\textbf{Theory of the Firm}

\begin{align*}
x \in X & \subseteq \mathbb{R}^n, \text{ input-output plans} \\
\mathbf{p} & \in \mathbb{R}_+^k, \text{ price-lists} \\
\pi(x, p) & = p \cdot x, \text{ firm's profit} \\
\phi(p) & = x, \text{ a constant correspondence; production set isn't affected by prices.} \\
\mu(p) & = \{ x \in X \mid x \text{ max's } \pi \text{ on } X \} \\
V(p) & = \max_{x \in X} \pi(x, p) = \max_{x \in X} p \cdot x, \text{ firm's profit as function of } p.
\end{align*}

The Maximum Theorem tells us that the firm's supply/demand correspondence \( \mu \) is nonempty-valued, compact-valued, UHC, and closed if the production set \( X \) is compact. Since it typically isn't, we need to restrict it to a compact set when applying the Maximum Theorem.
The Maximum Theorem

In Our Growth Theory Example

Recall that we wanted to establish the existence of a function $v: \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies

(*) $\forall x \in \mathbb{R}^+: v(x) = \max_{2 \in \mathbb{R}^+} \left[ u(f(x) - 2) + \beta v(2) \right] \text{ s.t. } 0 \leq 2 \leq f(x)$.

Note that $v$ — the same function $v$ — is on both sides of this (functional) equation. For any $x \in \mathbb{R}^+$, $v(x)$ was the value, at an arbitrary time $t$, of having capital stock $x$ — the present value at $t$ of the current and future stream of period-by-period values.

In order to show that there is such a $v$, we described $v(\cdot)$ as the fixed point of a transformation $T: F \to F$ that maps functions $v \in F$ (where $F$ is a set of functions $v: \mathbb{R}^+ \to \mathbb{R}^+$) into other functions, say $\tilde{v} \in F$. We defined the transformation $T$ (or equivalently, the function $\tilde{v}$ for a given $v$) as follows:

\[ \tilde{v}(x) = \max_{2 \in \mathbb{R}^+} \left[ u(f(x) - 2) + \beta v(2) \right] \text{ s.t. } 0 \leq 2 \leq f(x). \]

Clearly, a function $v: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (*) if and only if it's a fixed point of $T$, if and only if $\tilde{v} = v$. 
We want to establish that \( T: F \rightarrow F \) has a fixed point, where \( F \) is the set of all functions \( v: \mathbb{R^+} \rightarrow \mathbb{R} \). We want to show there is some function \( v: \mathbb{R^+} \rightarrow \mathbb{R} \) for which \( \tilde{v} = v \). We'll use a fixed point theorem to establish this, but we don't have a theorem that applies to the set \( F \) of all functions from \( \mathbb{R^+} \) into \( \mathbb{R} \). So we need to narrow down the set \( F \). If we can narrow it down to a set \( F' \subseteq F \) for which a fixed point theorem does apply, we'll have established that \( F' \) (and therefore \( F \)) does have a fixed point.

What set \( F' \) should we use? If we want to use the Banach fixed point theorem, for example, \( F' \) will have to satisfy the following conditions:

1. \( F' \) is a complete metric space (in some metric)
2. if \( v, w \in F' \), then \( v, w \in F' \) — i.e., \( T \) maps \( F' \) into \( F' \).
3. \( T \) is a contraction on \( F' \) — i.e., in the metric \( d \):
   \[ d(\tilde{v}, \tilde{w}) \leq \beta d(v, w) \]

We use \( C_b(\mathbb{R}^+) \), the set of all bounded continuous functions on \( \mathbb{R}^+ \), and the sup-metric on \( C_b(\mathbb{R}^+) \):

1. is easy to show.
2. is relatively straightforward. (We won't do it here.)
3. \( \tilde{v} \) is bounded is easy to show.
4. \( \tilde{v} \) is continuous: we'll do this here.
We show that

\((***)\) \( \forall \mathbf{g} \in C_b(\mathbb{R}^+), \tilde{\mathbf{v}} \text{ is continuous.} \)

**Maximum Theorem**

\[
x \in \mathbb{R}^l, \quad z \in \mathbb{R}^n
\]

**Growth Theory Application**

\[
e \in \mathbb{R}^m, \quad x \in \mathbb{R}^n
\]

\[
u : X \times E \to \mathbb{R}, \quad \tilde{u}(z, x) = u(f(x), z) + \varphi(v(z))
\]

\[
\Phi : E \to X, \quad \Phi(x) = [0, f(x)]
\]

\[
\mu : E \to X, \quad \mu(x) = \left\{ z \in \mathbb{R}^n \mid \tilde{u} \text{ is a solution} \right\}
\]

\[
\nabla : E \to \mathbb{R}, \quad \nabla(x)
\]

\[
\mu(x) = \arg\max_{z \in \Phi(x)} \tilde{u}(z, x)
\]

\[
(\text{we assume } \forall \mathbf{g} \in C(\mathbb{R}^+))
\]

\(\tilde{u}\) is clearly continuous if \(\tilde{f}\) and \(\tilde{u}\) are continuous.

Easy to show that \(\Phi\) is continuous (UHC & LHC).

\(\Phi\) is obviously compact-valued.

\[
\mu \text{ is UHC and } \tilde{v} \text{ is continuous.}
\]

Thus is what \((***)\)

we need: \((***)\) is satisfied

Therefore the contraction mapping theorem applies, and there is a fixed point of \(T\)

A \(v : \mathbb{R}^+ \to \mathbb{R}^n\) that satisfies \((*)\).