Let $S := \{ p \in \mathbb{R}^2_+ \mid p_1 + p_2 = 1 \}$, the unit-simplex in $\mathbb{R}^2$, and define the budget-set correspondence $B : S \rightarrow \mathbb{R}^2_+$ as follows:

$$B(p) = \{ x \in \mathbb{R}^2_+ \mid p \cdot x \leq p \cdot \hat{x} \}$$

Assume that $\hat{x} \in \mathbb{R}^2_{++}$.

(a) Prove that $B$ is closed (i.e., that it has a closed graph).

(b) Prove that $B$ is lower-hemicontinuous.

(c) Prove that $B$ is upper-hemicontinuous. (This is more difficult than (a) and (b).)

Suggestions for Part (c)

1. Let’s simplify by defining $\overline{B} : [0, 1] \rightarrow \mathbb{R}^2_+$ as $\overline{B}(p) := B(p, 1 - p)$. We’ll show that $\overline{B}$ is UHC on $[0, 1]$.

2. First consider any $\overline{p} \in (0, 1)$ — i.e., any $\overline{p}$ other than 0 or 1. Let $a$ and $b$ be two real numbers that satisfy $0 < a < \overline{p} < b < 1$ — in particular, $\overline{p} \in [a, b]$. You’ve shown in (a) that $B$ is closed on $S$, so it’s clearly closed on any closed subset of $S$, and $\overline{B}$ is therefore closed on $[a, b]$. Moreover, the range of $\overline{B}$ is clearly bounded on $[a, b]$. (Getting this range to be bounded was the reason for choosing $a > 0$ and $b < 1$.) Therefore $\overline{B}$ is closed and maps into a compact set, so it is UHC on $[a, b]$, by a theorem in the lecture notes. In particular, it is UHC at $\overline{p}$, and $\overline{p}$ was an arbitrary element of $(0, 1)$, so we’ve shown that $\overline{B}$ is UHC on $(0, 1)$.

Now we only need to show that $\overline{B}$ is UHC at $\overline{p} = 0$ and at $\overline{p} = 1$. Wlog, we do so for $\overline{p} = 0$.

Suppose that $\overline{B}$ is not UHC at $\overline{p} = 0$. Then there is an open set $V \subseteq \mathbb{R}^2_+$ such that $\overline{B}(\overline{p}) \subseteq V$ and a sequence $\{p_n\}$ in $[0, 1]$ such that $p_n \rightarrow \overline{p} = 0$ and for every $n \in \mathbb{N}$ we have $\overline{B}(p_n) \notin V$ — i.e., for each $n \in \mathbb{N}$ there is a bundle $x^n \in \mathbb{R}^2_+$ such that $x^n \in \overline{B}(p_n)$ and $x^n \notin V$.

3. First, suppose $x^n_1 > \hat{x}_1$ for some $n$. Then $x^n_2 < \hat{x}_2$ (why?), and it’s easy to show that $x^n \notin \overline{B}(\overline{p}) \subseteq V$, a contradiction. Therefore $x^n_1 \leq \hat{x}_1$ for every $n \in \mathbb{N}$.

4. Now we’re left with just the case $x^n_1 \leq \hat{x}_1$ for every $n \in \mathbb{N}$. Here you should be able to show that the sequence $\{x^n\}$ is bounded, so it has a convergent subsequence; write $\overline{x}$ for the limit of the subsequence. It’s easy to see that $\overline{x}_2 = \hat{x}_2$, from which you can show that $\overline{x} \in \overline{B}(\overline{p}) \subseteq V$, which you can again show yields a contradiction.