Economics 519: 2016 Exercise Set #6

1. Let \( X \) be the three-element set \( \{a, b, c\} \), and let \( d : X \times X \rightarrow \mathbb{R}_+ \) be the discrete metric:

\[
d(x, x') := \begin{cases} 
0 & \text{if } x = x' \\
1 & \text{otherwise}.
\end{cases}
\]

(a) Determine all the open sets in \((X, d)\) and all the closed sets in \((X, d)\).

(b) What if \( X = \mathbb{R} \) instead of \( \{a, b, c\} \), but still with the discrete metric \( d \) — what are the open sets and the closed sets? What if \( X \) is an arbitrary set with the discrete metric?

2. Recall from Exercise Set #4 that two norms \( \| \cdot \|_a \) and \( \| \cdot \|_b \) on a given vector space \( V \) are said to be equivalent if there exist positive numbers \( m \) and \( M \) such that

\[
\forall x \in V : m\|x\|_a \leq \|x\|_b \leq M\|x\|_a.
\]

But we haven’t said anything yet about why equivalence of norms might be a useful concept. An important (and obviously very useful) property of equivalent norms is that they generate the same open sets. Therefore when we’re considering convergence of sequences, or continuity of functions, etc., in normed vector spaces, and there are alternative but equivalent norms, it doesn’t matter which one we use. If one norm works better for our purpose, we can use that one.

Let’s prove that equivalent norms on \( \mathbb{R}^n \) generate the same open sets. Specifically, let’s say that a subset \( S \) of \( \mathbb{R}^n \) is open according to the norm \( \| \cdot \| \) if for every \( \overline{x} \in S \) there is an \( \epsilon > 0 \) such that \( \|x - \overline{x}\| < \epsilon \Rightarrow x \in S \). In other words, a set is open according to \( \| \cdot \| \) if it contains an \( \epsilon \)-ball in norm \( \| \cdot \| \) about each of its points \( \overline{x} \). The sets in \( \mathbb{R}^n \) we’ve previously defined as open are the ones that are open according to the Euclidean norm.

Prove that if \( \| \cdot \|_a \) and \( \| \cdot \|_b \) are equivalent norms on \( \mathbb{R}^n \) then a subset \( S \) of \( \mathbb{R}^n \) is open according to \( \| \cdot \|_a \) if and only if it is open according to \( \| \cdot \|_b \).

3. Here’s an immediate use of the fact in #2: Prove that if a sequence \( \{x(n)\} \) converges to \( \overline{x} \in \mathbb{R}^\ell \), then for each \( k \in \{1, \ldots, \ell\} \) the component sequence \( \{x_k(n)\} \) converges to \( \overline{x}_k \). According to #2 you can use whichever of the norms \( \| \cdot \|_1, \| \cdot \|_2, \) or \( \| \cdot \|_\infty \) that makes your proof the easiest, since we’ve already proved that these three norms are equivalent.
4. The Lexicographic Preference was introduced in the “Binary Relations” notes. Prove that for every bundle \( x \) in \( \mathbb{R}_+^2 \), the Lexicographic upper and lower contour sets of \( x \) (both strict and weak) are neither open nor closed. Note the contrast with Example 7 in the “Open and Closed Sets” lecture notes. **Note:** The weak upper contour set of a bundle \( \mathbf{x} \) is the set of bundles weakly preferred to \( \mathbf{x} \) — *i.e.*, the set \( \{ \mathbf{x} \in \mathbb{R}_+^2 \mid \mathbf{x} \succeq \mathbf{x} \} \). The strict upper contour set and the weak and strict lower contour sets are defined analogously.

5. For each of the sequences \( \{F_n\} \) below, from Exercise Set #5, determine whether whether the sequence \( \{F_n\} \) converges uniformly. If it does, determine how \( n \) can depend on \( \varepsilon \) in applying the definition of uniform convergence; if not, show that it’s not possible to define such an \( n \) for every \( \varepsilon > 0 \).

(a) For each \( n \in \mathbb{N} \), \( F_n : [0, 1] \to \mathbb{R} \) is the function

\[
F_n(x) = \begin{cases} 
\frac{n+1}{n} x, & \text{if } x \leq \frac{n}{n+1} \\
1, & \text{if } x \geq \frac{n}{n+1}.
\end{cases}
\]

(b) For each \( n \in \mathbb{N} \), \( F_n : [0, 1] \to \mathbb{R} \) is the function

\[
F_n(x) = \begin{cases} 
 nx, & \text{if } x \leq \frac{1}{n} \\
1, & \text{if } x \geq \frac{1}{n}.
\end{cases}
\]

6. The following remark appears in the lecture notes: If \( \{x_n\} \) converges to \( \mathbf{x} \) in a metric space \((X, d)\), then \( \mathbf{x} \) is the sequence’s only cluster point. Provide a proof of this remark.