1. Let $F : [0, 1] \to \mathbb{R}$ be the identity function $F(x) = x$, and for each $n \in \mathbb{N}$ let $F_n : [0, 1] \to \mathbb{R}$ be the function

$$F_n(x) = \begin{cases} \frac{n+1}{n} x, & \text{if } x \leq \frac{n}{n+1} \\ 1, & \text{if } x \geq \frac{n}{n+1}. \end{cases}$$

Note that each of these functions is in $C[0, 1]$, the set of continuous functions on $[0, 1]$. (Each function is the distribution function of a uniformly distributed random variable; more precisely, the restriction of the distribution function to the unit interval $[0, 1]$).

(a) Draw the graphs of $F, F_1, F_2, \text{ and } F_3$.

(b) For each $n \in \mathbb{N}$ and each $x \in [0, 1]$, determine the value of $|F_n(x) - F(x)|$.

(c) Determine whether the sequence $\{F_n\}$ converges pointwise. If it does, determine how $\bar{n}$ can depend on $x$ and $\epsilon$ in applying the definition of pointwise convergence; if not, show that it’s not possible to define such an $\bar{n}$ for every $x \in [0, 1]$ and $\epsilon > 0$.

(d) Determine whether the sequence $\{F_n\}$ converges in the metric space $\left(C[0, 1], \| \cdot \|_\infty \right)$.

2. For each $n \in \mathbb{N}$ let $F_n : [0, 1] \to \mathbb{R}$ be the function

$$F_n(x) = \begin{cases} nx, & \text{if } x \leq \frac{1}{n} \\ 1, & \text{if } x \geq \frac{1}{n}. \end{cases}$$

(a) Draw the graphs of $F_1, F_2, \text{ and } F_3$. Each of these functions is also the distribution function of a uniformly distributed random variable.

(b) Determine whether the sequence $\{F_n\}$ converges pointwise. If it does, determine how $\bar{n}$ can depend on $x$ and $\epsilon$ in applying the definition of pointwise convergence; if not, show that it’s not possible to define such an $\bar{n}$ for every $x \in [0, 1]$ and $\epsilon > 0$.

(c) Determine whether the sequence $\{F_n\}$ converges in the metric space $\left(C[0, 1], \| \cdot \|_\infty \right)$.

3. Let $V$ be a normed vector space. Prove that the only vector subspace of $V$ that’s bounded is the one-point space consisting of just the zero vector. (And therefore, in particular, the only normed vector space that’s bounded is the one-point space.)

4. The set of bounded real sequences, equipped with the sup-norm, is denoted by $\ell^\infty$. Verify that $\ell^\infty$ is normed vector space. You’ve already verified, in Set #4, that it’s a vector space, so you
needn’t do that again. And three of the defining properties of a norm, (N1), (N2), and (N4), are immediate from the definitions, so you needn’t prove those here. This leaves just one thing for you to prove: that the sup-norm, \( \|\{x_n\}\| = \sup\{|x_n| \mid n \in N\} \), satisfies the third property, (N3):

\[
\forall x, y \in \ell^\infty : \|x + y\| \leq \|x\| + \|y\|.
\]