1. If $R$ is a binary relation on a set $X$, an $R$-cycle is a list (an $n$-tuple) $(x_1, \ldots, x_n)$ of members of $X$ for which $x_kRx_{k+1}$ for $k = 1, \ldots, n - 1$ and $x_nRx_1$, and $R$ is said to be acyclic if there are no $R$-cycles. Note that if $xRx$ then $(x, x)$ is a cycle, so it follows that an acyclic relation must be irreflexive (according to this definition).

(a) Is a transitive relation necessarily acyclic? Is an acyclic relation necessarily transitive? Verify (i.e., prove) that your answers are correct.

(b) Prove that if $\succ$ is an acyclic relation on a set $X$, then every finite subset of $X$ has a $\succ$-maximal element.

(c) Define $\succsim$ on $\mathbb{R}_+^3$ by $x \succsim y$ if $x_i \geq y_i$ for at least two of the components, i.e., for at least two of $i = 1, 2, 3$. Is $\succsim$ complete? Is it transitive? Define $\succ$ in the usual way: $x \succ y$ if $x \succsim y$ but not $y \succsim x$. Is $\succ$ acyclic? Can you relate this exercise to the majority-rule procedure for aggregating preferences?

2. A firm uses three inputs to produce its single product, according to the production function $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ defined by $q = f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$. This question is about the $q$-level isoquant — the level set $f(x) = q$ in $\mathbb{R}_+^3$. Use the Implicit Function Theorem to obtain your answers.

(a) Determine whether you can express the $q$-level isoquant by a differentiable function $g$ giving the $x_3$ input level as a function of the input levels $x_1$ and $x_2$: $x_3 = g(x_1, x_2)$.

(b) Determine the marginal rates of technical substitution (MRTS) between input 1 and input 3, and between input 2 and input 3 — i.e., the derivatives $\frac{\partial x_3}{\partial x_1}$ and $\frac{\partial x_3}{\partial x_2}$ on the $q$-isoquant.

3. Do the exercise on the following page on just-noticeable differences (JNDs). The exercise refers to an exercise in Kreps’s book; that exercise from Kreps is attached. (You’re not to do the Kreps exercise; it’s attached in order for you to be able to do the JND exercise.)
Exercise on Just Noticeable Differences

It’s been known since the mid-1800s that for many sensory phenomena people can detect differences or changes in intensity only if the difference is sufficiently large. The size of the required difference is called a just noticeable difference, or JND — also often written in lower-case: jnd. Examples are brightness of lights; the loudness and the pitch of tones; the weight and the length of objects; the saltiness and sweetness of foods; the degree of compression in digital images; and many other sensory phenomena. This fact has obvious importance for many fields, such as marketing and computer graphics, to name just two. And it has implications for the theory of choice: it would seem to mean (informally speaking) that preferences may be “incomplete.”

Suppose that the intensity of some sensory phenomenon — say, the sweetness of a cup of coffee or a bowl of cereal — can be represented by non-negative real numbers $x \in \mathbb{R}^+$, namely the number of grains of sugar in the cup or the bowl. Suppose the size of the relevant JND is $m$ — some person can detect a change in sweetness only if more than $m$ grains are added or removed. It’s natural to represent this by a strict preorder on $\mathbb{R}_+$:

$$x' \succ x \text{ if } x' > x + m. \quad (\ast)$$

The preorder doesn’t necessarily describe the person’s preference (although it could): instead, $x' \succ x$ simply means that this person can tell that $x'$ is larger (sweeter) than $x$. (JNDs $\Delta x$ actually behave according to Weber’s Law, $\Delta x = kx$, where the constant $k$ varies greatly for different phenomena and much less so across different people. We’ll use the simpler linear version of the law, above, in which the JND is constant: $\Delta x = m$.)

(a) In Kreps’s Exercise #1.17, define $\succ$ as in $(\ast)$ above. Provide a definition of the relation $\sim$ (which Kreps calls “positive indifference”) for which $\succ$ and $\sim$ satisfy the assumptions (1) - (4) in #1.17. Verify that (1) - (4) are satisfied. In other words, verify that this definition of $\succ$ and $\sim$ on $\mathbb{R}_+$ constitute a special case of the model Kreps develops in #1.17. (Kreps restricts the model to finite sets $X$; the special case in this exercise doesn’t require that $X$ be finite.)

(b) In parts (a)-(f) of #1.17 Kreps asks you to assume that $\succ$ and $\sim$ are arbitrary relations satisfying assumptions (1)-(4). In our exercise here, they’re specific relations. Answer parts (a), (b), (d), and (e) of #1.17 just for the relations $\succ$ and $\sim$ defined here, by working directly with $\succ$ and $\sim$ as defined here. (For (e), where Kreps says “the answer is no, in general,” the answer is actually yes for this special case.)
As an alternative to the model in #1.17, define $\succsim$ as in (⋆) above, but define $\sim$ by

$$x' \sim x \text{ if neither } x' \succ x \text{ nor } x \succ x'. \quad (**)$$

(c) Do $\succ$ and $\sim$ satisfy assumptions (1)-(4)? Does there exist any function $U : \mathbb{R}_+ \to \mathbb{R}$ such that $x' \succ x \iff U(x') > U(x)$? Verify your answer.

(d) Kreps calls his $\sim$ “positive indifference.” In the alternative model described here, suppose that the decision-maker always prefers a larger (sweeter) $x$ if she can distinguish that it’s larger. How would you describe what $\sim$ means about her preferences in this alternative model? While both models are models of incomplete preference in the informal sense — in fact, they’re both modeling the exact same phenomenon, in this example — you’ve shown that the formal properties of the relations $\sim$ and $\succsim$ (completeness, transitivity, etc.) are different in the two models. Does either model seem to be a more intuitive, or more complete, description of the situation?

**Historical note:**

Experimental psychology began in the mid-1800s with the research of Ernst Weber and Gustav Fechner, who studied the ability of people to detect differences in sensory phenomena. Weber’s experiments led him to conclude that people could detect the difference in two objects’ weight only when the difference was sufficiently large. He postulated what Fechner subsequently named Weber’s Law, described above. Fechner investigated people’s ability to discriminate between the loudness of various sounds, and found that Weber’s Law held there as well. Discrimination doesn’t follow the law exactly: a given person’s discrimination varies slightly from one trial to another. The convention that’s used is to say a person can distinguish two phenomena if she can distinguish them on at least half the trials she’s exposed to.
1.17. In this problem, we consider an alternative theory to the standard model, in which the consumer is unable/unwilling to make certain preference judgments. We desire a theory along the following lines: There are two primitive relations that the consumer provides, strict preference $\succ$ and positive indifference $\sim$. The following properties are held to be desirable in this theory:

1. $\succ$ is asymmetric and transitive;

2. $\sim$ is reflexive, symmetric, and transitive;

3. if $x \succ y$ and $y \sim z$, then $x \succ z$; and

4. if $x \sim y$ and $y \succ z$, then $x \succ z$.

For all parts of this problem, assume that $X$, the set on which $\succ$ and $\sim$ are defined, is a finite set.

(a) Prove that 1 through 4 imply: If $x \succ y$, then neither $y \sim x$ nor $x \sim y$.

(b) Given $\succ$ and $\sim$ (defined for a finite set $X$) with the four properties listed, construct a weak preference relationship $\succeq$ by $x \succeq y$ if $x \succ y$ or $x \sim y$. Is this weak preference relationship complete? Is it transitive?

(c) Suppose we begin with a primitive weak preference relationship $\succeq$ and define $\succ$ and $\sim$ from it in the usual manner: $x \succ y$ if $x \succeq y$ and not $y \succeq x$, and $x \sim y$ if $x \succeq y$ and $y \succeq x$. What properties must $\succeq$ have so that $\succ$ and $\sim$ so defined have properties 1 through 4?

(d) Suppose we have a function $U : X \rightarrow R$ and we define $x \succ y$ if $U(x) > U(y) + 1$ and $x \sim y$ if $U(x) = U(y)$. That is, indifferent bundles have the same utility; to get strict preference, there must be a "large enough" utility difference between the two bundles. Do $\succ$ and $\sim$ so constructed from $U$ have any/all of the properties 1 through 4?

(e) Suppose we have $\succ$ and $\sim$ satisfying 1 through 4 for a finite set $X$. Does there exist a function $U : X \rightarrow R$ such that $U(x) = U(y)$ if and only if $x \sim y$ and $U(x) > U(y) + 1$ if and only if $x \succ y$? To save you the effort of trying to prove this, I will tell you that the answer is no, in general. Provide a counterexample.

(f) (Good luck.) Can you devise an additional property or properties for $\succ$ and $\sim$ such that we get precisely the sort of numerical representation described in part d? (This is quite difficult; you may want to ask your instructor for a hint.)