**Intertemporal Exercise: Solution**

\[ u(x_0, x_1) = x_0^3 x_1^2 \quad \text{MRS} = \left(\frac{2}{3}\right) \frac{x_1}{x_0} \]

\[
f(\bar{z}) = \begin{cases} 
4z - \frac{1}{8} \bar{z}^2 & \bar{z} \leq 16 \\
32 & \bar{z} > 16
\end{cases}
\]

\[
f'(\bar{z}) = \begin{cases} 
4 - \frac{1}{4} \bar{z} & \bar{z} \leq 16 \\
0 & \bar{z} > 16
\end{cases}
\]

(a) \( \max \ u(x_0, x_1) + f(\bar{z}) \)

\[
\frac{du}{d\bar{z}} \leq 0 \quad \text{and} \quad \frac{du}{d\bar{z}} = 0 \quad \iff \quad f'(\bar{z}) = \frac{u_0}{u_1},
\]

\( \text{i.e., MRT = MRS.} \)

At \( \bar{z} = 8 \):
\[
f(8) = 24, \quad x_0 = 38 - 8 = 30, \quad x_1 = 16 + 24 = 40.
\]

MRS = \( \left(\frac{2}{3}\right) \frac{40}{30} = 2 \), MRT = 4 - 2 = 2. See Figure 1.

(b) \( r = 100\% = 1 \): \( 1 + r = 2, \frac{1}{1+r} = \frac{1}{2} \)

The plans in (a) satisfy MRS = 2 = 1 + r and MRT = 2 = 1 + r;

\( \therefore \) That's the investment plan that maximizes NPV, and the consumption plan maximizes \( u(\ldots) \) among the bundles with the same NPV. See Figure 2.

Therefore \( \bar{z} = 8, \quad f(8) = 24, \quad x_0 = 30, \quad x_1 = 40 \), and Benjamin neither borrows nor lends.

\[
\begin{align*}
V_0(38, 16) &= 38 + \frac{1}{2} (16) = 46 \\
V_0(-8, 24) &= -8 + \frac{1}{2} (24) = 4 \\
V_0(30, 40) &= 30 + \frac{1}{2} (40) = 50.
\end{align*}
\]
(c) \( r = 200\%: \quad 1 + r = 3, \quad \frac{1}{1+r} = \frac{1}{3}. \)

\[ \begin{align*}
V_o(38, 16) &= 38 + \frac{1}{3}(16) = 43 \frac{3}{3} & \text{(38, 16) and (33, 40) are on the same NPV-contour.} \\
V_o(-8, 24) &= -8 + \frac{1}{3}(24) = 0 \\
V_o(30, 40) &= 30 + \frac{1}{3}(40) = 43 \frac{3}{3}
\end{align*} \]

See Figure 3: It's clear from the diagram that Benjamin will choose a smaller \( t \) will save, will consume less at \( t = 0 \) and more at \( t = 1 \) and will be better off (on a higher indifference curve).

(d) At \( r = 200\%: \)

\[ f'(z) = 1 + r : \quad 4 - \frac{1}{4}z = 3 \quad \text{i.e.,} \quad z = 4; \quad f(z) = 14. \]

\[ \begin{align*}
V_o(-4, 14) &= -4 + \frac{1}{3}(14) = 2 \frac{2}{3} \\
V_o(38, 16) &= 38 + \frac{1}{3}(16) = 43 \frac{3}{3}
\end{align*} \]

\[ W = V_o(38, 16) + V_o(-4, 14) = 43 \frac{3}{3} + 2 \frac{2}{3} = 44. \]

\[ \max x_0(x_0, x_1) \quad \text{s.t.} \quad x_0 + \frac{1}{14}x_1 = W = 44. \]

Solution:

\[ (\frac{3}{2}) \frac{x_0}{x_1} = 1 + r = 3 \quad \text{and} \quad x_0 + \frac{1}{3}x_1 = 44 \quad \text{(BC)} \]

\[ x_1 = 2x_0 \quad \text{Substituting into (BC):} \quad x_0 + \frac{1}{3}(2x_0) = 44 \]

\[ \text{i.e.,} \quad x_0 + \frac{5}{3}x_0 = 44 \]

\[ \text{i.e.,} \quad \frac{8}{3}x_0 = 44 \]

\[ \text{i.e.,} \quad x_0 = \frac{132}{5} = 26.4 \]

\[ x_1 = 52.8 \]

Check: \( (\frac{3}{2})(\frac{52.8}{26.4}) = 3 \)

\[ 26.4 + \frac{1}{3}(52.8) = 26.4 + 17.6 = 44. \]

\[ \therefore z = 4, \quad f'(z) = 14; \quad x_0 = 26.4, \quad x_1 = 52.8; \]

Benjamin saves \( x_0 - z - x_o = 38 - 4 - 26.4 = 7.6 = 5 \)

And receives \((1 + r)s = (3)(7.6) = 22.8 \) at \( t = 1 \)

So \( x_1 = x_0 + f(z) + (1 + r)s = 16 + 14 + 22.8 = 52.8. \)
Figure 1

Figure 2

Figure 3