P is the price ratio $\frac{P_x}{P_y}$.

(a) Each girl's demand: $y_G = ax$ and $px + y_G = 8$.

\[ \therefore (a + p)x_G = 8 \quad \therefore x_G = \frac{8}{a + p} \quad y_G = \frac{a}{a + p} \cdot 8. \]

Each boy's demand:

\[ x_B = \begin{cases} 8, & \text{if } p < 1 \\ \left[ \frac{a}{q}, 8 \right], & \text{if } p = 1 \\ 0, & \text{if } p > 1 \end{cases} \quad y_B = \begin{cases} 0, & \text{if } p < 1 \\ \left[ \frac{a}{q}, 8 \right], & \text{if } p = 1 \\ 8p, & \text{if } p > 1 \end{cases} \]

Excess demand for honey ($q = \frac{P_x}{P_y}$):

\[ \begin{align*}
\text{If } q < 1 & : (\frac{aq}{1+aq} - \frac{8}{1+aq} \quad r = \frac{8}{1+aq} \\
\text{If } q > 1 & : (\frac{aq}{1+aq} - \frac{8}{1+aq} \quad r = 8r \frac{aq}{1+aq} - 1) = 8r \frac{aq}{1+aq} \\
\text{If } q = 1 & : \text{Each boy will accept any bundle s.t.} \end{align*} \]

Equilibrium:

\[ \begin{align*}
\text{If } q < 1 & : \text{Excess demand for honey.} \\
\text{If } q > 1 & : \text{Excess supply of honey.} \\
\text{If } q = 1 & : \text{Each boy will accept any bundle s.t.} \\
\end{align*} \]

\[ \begin{align*}
x_B + x_G &= 8 - \frac{8}{1+aq} + \frac{a}{1+aq} = \frac{8aq}{1+aq} \\
y_B + y_G &= \frac{8}{1+aq} + \frac{a}{1+aq} = \frac{8}{1+aq} \\
\end{align*} \]

So that $x_G + x_B = (\frac{8}{1+aq} - \frac{8}{1+aq} + \frac{a}{1+aq}) = 8r$ and

$y_G + y_B = (\frac{8}{1+aq} + \frac{a}{1+aq}) = 8r$, meaning both markets.
(b) The Pareto optimal allocations are the ones that satisfy:

1. \[ \sum_{i=1}^{r} x_{bi} + \sum_{i=1}^{r} x_{gi} = 8r \]
2. \[ \sum_{i=1}^{r} y_{bi} + \sum_{i=1}^{r} y_{gi} = 8r \]
3. \[ \forall i: y_{gi} = ax_{gi} \]

(If \( y_{gi} > ax_{gi} \) for some \( i \), she could give \( y_{gi} - ax_{gi} \) pints of honey to boys, making them better off but not making her worse off. Similarly if \( y_{gi} < ax_{gi} \).)

If (1), (2), (3) satisfied:

A girl's utility can be increased only by giving her more of both goods, clearly hurting someone else.

A boy's utility can be increased only by:

(i) reducing some \( x_{gi} \) or \( y_{gi} \), which reduces \( y_{gi} \) or
(ii) reallocating among boys, which cannot be a Pareto improvement because boys care only about the total \( x_{bi} + y_{bi} \).
(c) \( r = 1 \):

The core is the lower left half of the main diagonal. Only the diagonal allocations are Pareto optimal with \( r = 1 \) (in (b)). At their endowment \( u_G = 0 \) and \( u_B = 8 \).

If \( r \geq 2 \):