Game Forms and Mechanism Design

Recall that a **game** is an $n$-tuple $(S_i, \pi_i)_{i=1}^n$, where

$S_i$ is $i$’s strategy or action set ($i = 1, \ldots, n$),

$\pi_i : S_1 \times \cdots \times S_n \rightarrow \mathbb{R}$ is $i$’s payoff function ($i = 1, \ldots, n$).

A **game form** is a way to model the rules of a game, or an institution, independently of the players’ utility functions over the game’s outcomes. The notion of a game form is an important idea for **mechanism design** (also called **institution design** or **market design**).

**Definition:** Let $X$ be a set of possible outcomes. A **game form** for $X$ consists of

1. $n$ action sets $S_1, \ldots, S_n$ , and
2. an outcome function $\varphi : S_1 \times \cdots \times S_n \rightarrow X$.

**Definition:** Given an outcomes set $X$ and

1. a game form $(S_1, \ldots, S_n; \varphi)$ for $X$, and
2. $n$ utility functions $u_i : X \rightarrow \mathbb{R}$ over outcomes ($i = 1, \ldots, n$),

the **associated game** or **induced game** is defined by the $n$ action sets $S_1, \ldots, S_n$ and the $n$ payoff functions

$$\tilde{u}_i(s_1, \ldots, s_n) := u_i(\varphi(s_1, \ldots, s_n)), \; i = 1, \ldots, n.$$ 

In our public goods model, where $x$ is the level at which the public good is provided and $y_i$ is the number of dollars $i$ spends on other goods, an outcome is an $(n+1)$-tuple $(x, y_1, \ldots, y_n) \in \mathbb{R}_+^{n+1}$, so our outcome set is $X = \mathbb{R}_+^{n+1}$. Assume that the cost of the public good is given by $C(x) = cx$, so marginal cost is $c$ (for example, $c$ is the price that’s charged for each unit of the public good).

**Example:** The Voluntary Contributions Mechanism (VCM) for a public good.

The VCM institution, or game form, is defined by the following action sets and outcome function:

**Actions:** Each person $i$ chooses a contribution $m_i$ in the action set $\mathbb{R}_+$. Let $\mathbf{m} = (m_1, \ldots, m_n)$.

**Outcome function:**

$x = \pi(\mathbf{m}) = \frac{1}{c} \sum_{i=1}^n m_i \quad (i.e., \; x \; is \; whatever \; quantity \; the \; contributions \; \sum_{i=1}^n m_i \; will \; buy);$ 

$y_i = \hat{y}_i - t_i$, where $t_i = \tau^i(\mathbf{m}) = m_i \quad (i.e., \; i$’s “tax” is simply his contribution, $m_i$).

Thus, the outcome function is $\varphi(\mathbf{m}) = (\pi(\mathbf{m}), \hat{y}_1 - \tau^1(\mathbf{m}), \ldots, \hat{y}_n - \tau^n(\mathbf{m})).$

The induced game is given by the utility functions $u^i(x, y_i), \; i = 1, \ldots, n$, so the payoff functions in the induced game are

$$\tilde{u}^i(m_1, \ldots, m_n) := u^i(\pi(\mathbf{m}), \hat{y}_i - \tau^i(\mathbf{m})) = u^i\left(\frac{1}{c} \sum_{j=1}^n m_j, \; \hat{y}_i - m_i\right), \; i = 1, \ldots, n.$$
The Nash equilibrium of the VCM institution (i.e., the NE of the associated game) is as follows:

The first-order marginal condition that characterizes individual $i$’s choice of $m_i$ is

\[ \frac{\partial \tilde{u}_i}{\partial m_i} \leq 0 \quad \text{and} \quad \frac{\partial \tilde{u}_i}{\partial m_i} = 0 \text{ if } m_i > 0. \]

We have

\[ \frac{\partial \tilde{u}_i}{\partial m_i} = \frac{\partial u^i}{\partial x_i} \frac{\partial \pi}{\partial m_i} + \frac{\partial u^i}{\partial y_i} \frac{\partial (-\tau^i)}{\partial m_i} = \frac{1}{c} u^i_x - u^i_y. \]

Therefore

\[ \frac{\partial \tilde{u}_i}{\partial m_i} \leq 0 \quad \text{if and only if} \quad \frac{u^i_x}{u^i_y} \leq c. \]

Therefore the FOMC above, for individual $i$, can be written as

\[ \frac{u^i_x}{u^i_y} \leq c \quad \text{and} \quad \frac{u^i_x}{u^i_y} = c \text{ if } m_i > 0 \]

\[ \text{i.e.,} \quad \text{MRS}_i \leq MC \quad \text{and} \quad \text{MRS}_i = MC \text{ if } m_i > 0. \]

Note that this is identical to the market outcome we obtained earlier, in which the public good is provided at a level that’s less than the Pareto level: those who contribute are only those with the largest MRS; everyone else is a free rider; and no one will contribute if everyone has $\text{MRS}_i < MC$ when $x = 0$.

**Mechanism Design:** The mechanism design problem is to devise an outcome function $\varphi$ for which the Nash equilibria (or some other specified solution) have one or more desirable properties — for example, an outcome function for which the Nash equilibria are Pareto efficient. For our simple public-goods model, the outcome function $\varphi$ is the $(n+1)$-tuple of functions $(\pi, \tau^1, \ldots, \tau^n)$, so our mechanism design problem is to devise a provision function $\pi$ and tax/transfer functions $\tau^i$ for each $i$ for which the Nash equilibrium is Pareto efficient, or better yet, is a Lindahl equilibrium allocation.

The first institution/mechanism with Pareto efficient Nash equilibria was devised by Grove & Ledyard. The first mechanism with Lindahl Nash equilibria was devised by Leo Hurwicz.