Arrow’s Walrasian Model
of Public Goods and Other Externalities

Arrow showed that we can recast the public-goods allocation problem as one involving only private goods, so that our Walrasian analysis applies. Arrow defined each individual’s consumption of the public good as a distinct commodity, with a distinct market and price, but with “jointness” in the production of these goods. Here’s how this works in our one-public-good-one-private-good model with \( n \) consumers (where \( X \) is the public good and \( Y \) is the private good):

We redefine the economy as having \( n + 1 \) goods \( X_1, \ldots, X_n, Y \), with quantities denoted by \( x_1, \ldots, x_n, y \). An allocation is therefore an \( n(n + 1) \)-tuple

\[
\left((x_1^1, \ldots, x_n^1, y^1), (x_1^2, \ldots, x_n^2, y^2), \ldots, (x_1^n, \ldots, x_n^n, y^n)\right) \in \mathbb{R}^{n(n+1)}.
\]

However, both the production possibilities and the consumption possibilities in this economy are assumed to have a special character:

1. The \( X \)-goods are “joint products” in any firm’s production process: A production plan for a firm is an \((n + 1)\)-tuple \((z, q) = (z, q_1, \ldots, q_n) \in \mathbb{R}^{n+1}_+\), where \( z \) is the amount of the private good the firm uses as input and \( q_i \) is the output of commodity \( X_i \), but the firm has the technological constraint \( q_1 = q_2 = \cdots = q_n \). This is exactly like the classical joint products mutton and wool that are produced by raising sheep.

2. Consumer \( i \)’s consumption set is \( \{ (x_1^i, \ldots, x_n^i, y^i) \in \mathbb{R}^{n+1}_+ \mid j \neq i \Rightarrow x^i_j = 0 \} \) — i.e., Consumer \( i \) can consume only the goods \( X_i \) and \( Y \). So while Consumer \( i \)’s utility function \( u^i \) is technically defined on the domain \( \mathbb{R}^{n+1}_+ \), we can more intuitively write \( u^i \) as defined on bundles \((x^i, y^i) \in \mathbb{R}^2_+\). Therefore we can simplify the notation, defining an allocation to consumers as a \( 2n \)-tuple \((x_i, y_i) \in \mathbb{R}^{2n}_+\).

Now a Lindahl equilibrium is just a Walrasian equilibrium of this joint-product economy. Specifically (and assuming for simplicity that there is just a single producer/firm, which is a price-taker), a Walrasian equilibrium is a price-list \((\hat{p}_1, \ldots, \hat{p}_n, \hat{p}_y) \in \mathbb{R}^{n+1}_+\), a consumption allocation \((\hat{x}_i, \hat{y}_i) \in \mathbb{R}^{2n}_+\) and a production plan \((\hat{z}, \hat{q}_1, \ldots, \hat{q}_n) \in \mathbb{R}^{n+1}_+\) that satisfy

\begin{align*}
(U\text{-max}) & \quad \forall i : (\hat{x}_i, \hat{y}_i) \text{ maximizes } u^i(x_i, y_i) \text{ subject to } \hat{p}_i x_i + y_i \leq \hat{y}_i + \theta_i \pi(\hat{z}, \hat{q}) \\
(\pi\text{-max}) & \quad (\hat{z}, \hat{q}) \text{ maximizes } \pi(z, q_1, \ldots, q_n) = \sum_{i=1}^n \hat{p}_i q_i - \hat{p}_y z \text{ subject to } q_1 = \cdots = q_n = f(z) \\
(M\text{-Clr}) & \quad \forall i : \hat{x}_i = \hat{q}_i \quad \text{ and } \quad \hat{z} + \sum_{i=1}^n \hat{y}_i \leq \sum_{i=1}^n y_i, \text{ with equality if } \hat{p}_y > 0.
\end{align*}
Therefore the First Welfare Theorem applies: if the utility functions and production functions satisfy the usual assumptions, then the equilibrium allocation will be Pareto efficient.

But Arrow’s model also makes it clear that the Walrasian models’s price-taking assumption for consumers is unrealistic here: for each of the distinct goods $X_i$ there is only one person on the demand side of the market. The only person who cares about the good $X_i$ is person $i$. It’s clearly unrealistic to assume that any of the participants will take their own price (or Lindahl cost share) as given. This was Arrow’s motivation for modeling things this way — to clarify this point.