Arrow’s Pricing Formula for Securities

Let $S$ be a finite set of states of the world and let $N$ be the index set for a finite set of consumers. Generic elements of $S$ and $N$ are denoted by $s \in S$ and $i \in N$. In a slight abuse of notation, we also use $S$ and $N$ to denote the number of elements in $S$ and $N$. We assume that there is only one good; each consumer $i \in N$ is endowed with $\tilde{x}_0^i$ units of the good today and with $\tilde{x}_s^i$ units in state $s$ tomorrow. It’s convenient to think of the good as dollars. Let $\tilde{x}^i = (\tilde{x}_0^i, \ldots, \tilde{x}_S^i)$. A consumption plan for consumer $i$ is a $(1 + S)$-tuple $(x_0^i, x^i) \in \mathbb{R}_+^{1+S}$, and an allocation is an $N$-tuple of plans, $(x_0^i, x^i)_{i \in N}$. Each consumer evaluates consumption plans according to a utility function $u^i : \mathbb{R}_+^{1+S} \to \mathbb{R}$. The economy is fully described by the set $S$ of states and by the $N$-tuple $(u^i, \tilde{x}_0^i, \tilde{x}^i)_{i \in N}$ of consumers.

A set of securities for this economy is an $S \times K$ matrix $D$. Each column of $D$ is the $S \times 1$ returns vector or dividends vector of one of the securities: the element $d_{sk}$ specifies how many dollars one unit of security $k$ will return tomorrow if state $s$ occurs. Note that $d_{sk}$ may be positive, zero, or negative. The $K$ columns of $D$ are thus the $K$ securities. Consumers purchase or sell units of the securities today and hold them until tomorrow, when one of the states $s \in S$ is realized and each security $k$ returns $d_{sk}$ dollars for every unit of the security a consumer owns. We denote by $y_k^i$ the number of units of security $k$ purchased by consumer $i$; $y_k^i$ may be positive, zero, or negative. Consumer $i$’s portfolio is the $K$-tuple $(y_1^i, \ldots, y_K^i)$, which we denote by $y^i$. Note that if consumer $i$ purchases the portfolio $y^i$, then his vector of state-contingent returns will be the $S$-tuple $Dy^i$. (It’s most convenient here to write $y^i$ and $Dy^i$ as $K \times 1$ and $S \times 1$ column vectors.) We denote the price of security $k$ by $q_k$, and we write $q = (q_1, \ldots, q_K)$.

**Definition:** An equilibrium of the securities markets defined by the matrix $D$ is a $(K + NK + N(1 + S))$-tuple $(q, (y^i)_{i \in N}, (x_0^i, x^i)_{i \in N}) \in \mathbb{R}_+^K \times \mathbb{R}^{NK} \times \mathbb{R}_+^{N(1+S)}$ that satisfies the utility-maximization and market-clearing conditions:

(U-M) \quad $\forall i \in N : (y^i, x_0^i, x^i)$ maximizes $u^i(x_0^i, x^i)$ subject to the constraints

\begin{align*}
x_0^i + q \cdot y^i & \leq \tilde{x}_0^i \quad \text{and} \\
x_s^i & \leq \tilde{x}_s^i + \sum_{k=1}^K d_{sk} y_k^i, \quad \forall s \in S, \quad \text{i.e.,} \quad x^i \leq \tilde{x}^i + Dy^i
\end{align*}

(M-C) \quad $\sum_{i=1}^N x_0^i = \sum_{i=1}^N \tilde{x}_0^i$ and $\sum_{i=1}^N y_k^i = 0, \quad k = 1, \ldots, K$.

**Examples:** Our “Extended Example of Equilibrium Under Uncertainty” contains several examples of securities markets using this model. Part 3 of the example is a market with a single security, a credit instrument such as a saving account or a bond. Part 4 adds a second security, an insurance contract.
**Example:** Suppose there are only two states, \( s = H \) and \( s = L \), and one security, which returns \( a \) in state \( H \) and \( b \) in state \( L \). By choosing \( y \), the number of units of the security he will buy at today’s security price \( q \), a consumer can vary \( x_H \) and \( x_L \), but not independently:

\[
\begin{bmatrix}
  x_H - \hat{x}_H \\
  x_L - \hat{x}_L
\end{bmatrix} =
\begin{bmatrix}
a \\
b
\end{bmatrix} y \quad \text{and} \quad x_0 = \hat{x}_0 - qy.
\]

Thus, giving up \( y \) units of consumption today will only allow him to augment his consumption tomorrow by multiples of \((a, b)\) across the two states.

Now suppose there’s a second security, which returns \( c \) in state \( H \) and \( d \) in state \( L \). If \((c, d)\) is a multiple of \((a, b)\), then nothing is gained by the introduction of the second security: choosing amounts \( y_1 \) and \( y_2 \) of the two securities still augments one’s consumption tomorrow only by multiples of \((a, b)\). But if \((a, b)\) and \((c, d)\) are not multiples of one another — i.e., if they’re linearly independent — then for any state-contingent consumptions \( x_H \) and \( x_L \) tomorrow, the equation

\[
\begin{bmatrix}
  x_H - \hat{x}_H \\
  x_L - \hat{x}_L
\end{bmatrix} =
\begin{bmatrix}
a & c \\
b & d
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
\]

has a solution \((y_1, y_2)\). Thus, in this case, state-contingent consumption tomorrow can be augmented by any amounts \( x_H - \hat{x}_H \) and \( x_L - \hat{x}_L \) by giving up some amount of consumption today in order to purchase some amounts \( y_1 \) and \( y_2 \) of the two securities. More securities would not add anything, but would not hurt either: as long as the securities returns matrix has two linearly independent columns (securities), any state-contingent consumptions can be achieved. More generally, with \( S \) states, the securities returns matrix \( D \) must have \( S \) linearly independent columns — i.e., we must have \( \text{rank} \ D = S \). We could equivalently say that the securities must \text{span} \ the space \( \mathbb{R}^S \).

It seems intuitive that this spanning condition will be necessary and sufficient to ensure that the securities markets achieve the same outcome as with complete Arrow-Debreu contingent claims markets — that an equilibrium allocation attained via securities markets will coincide with an Arrow-Debreu allocation. We now verify this intuition.

To simplify notation, let’s temporarily substitute \( z_0 \) for \( x_0 - \hat{x}_0 \) and \( z_s \) for each \( x_s - \hat{x}_s \). The key to establishing the equivalence of equilibrium outcomes is the individual consumer’s budget constraints: we show that if the securities span \( \mathbb{R}^S \), then both market structures present the consumer with exactly the same budget sets at their respective equilibrium prices. In our \( z \)-notation, the consumer’s Arrow-Debreu budget constraint is \( z_0 + p \cdot z = 0 \). We wish to be able to show that at some security prices \( q \) the constraint \( z_0 + q \cdot y = 0 \), together with the fact that \( z = Dy \), makes exactly the same set of \((z_0, z)\)’s available as the constraint \( z_0 + p \cdot z = 0 \) does.
The following proposition establishes that this is so if the securities span \( \mathbb{R}^S \) and if their prices are related to the contingent claims prices \( \mathbf{p} \) according to \( \mathbf{q} = \mathbf{p}D \). The proposition then leads to the subsequent theorem which establishes the equivalence between the securities markets equilibrium and the Arrow-Debreu equilibrium.

**Proposition:** Let \( \mathbf{p} \in \mathbb{R}^S \); let \( D \) be an \( S \times K \) matrix; let \( \mathbf{q} = \mathbf{p}D \in \mathbb{R}^K \); and let

\[
A = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid z_0 + \mathbf{p} \cdot \mathbf{z} = 0\} \quad \text{and} \\
B = \{(z_0, \mathbf{z}) \in \mathbb{R}^{1+S} \mid \exists \mathbf{y} \in \mathbb{R}^K : z_0 + \mathbf{q} \cdot \mathbf{y} = 0 \text{ and } \mathbf{z} = D\mathbf{y}\}.
\]

If \( \text{rank } D = S \), then \( A = B \).

**Proof:**

Note that if \( \mathbf{z} = D\mathbf{y} \) then \( \mathbf{p} \cdot \mathbf{z} = \mathbf{p} \cdot (D\mathbf{y}) = (\mathbf{p}D) \cdot \mathbf{y} = \mathbf{q} \cdot \mathbf{y} \). We show that \( A \subseteq B \) and \( B \subseteq A \).

(i) Let \( (z_0, \mathbf{z}) \in A \). Since \( \text{rank } D = S \), there is a \( \mathbf{y} \in \mathbb{R}^K \) that satisfies \( \mathbf{z} = D\mathbf{y} \). Since \( z_0 + \mathbf{p} \cdot \mathbf{z} = 0 \) (because \( (z_0, \mathbf{z}) \in A \) and \( \mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y} \) (because \( \mathbf{z} = D\mathbf{y} \)), we have \( z_0 + \mathbf{q} \cdot \mathbf{y} = 0 \), and therefore \( (z_0, \mathbf{z}) \in B \).

(ii) Let \( (z_0, \mathbf{z}) \in B \). Then, according to the definition of \( B \), there is a \( \mathbf{y} \in \mathbb{R}^K \) that satisfies both \( z_0 + \mathbf{q} \cdot \mathbf{y} = 0 \) and \( \mathbf{z} = D\mathbf{y} \). Therefore \( \mathbf{p} \cdot \mathbf{z} = \mathbf{q} \cdot \mathbf{y} \), and it follows that \( z_0 + \mathbf{p} \cdot \mathbf{z} = 0 \), and therefore \( (z_0, \mathbf{z}) \in A \).

**Theorem:** Let \( D \) be an \( S \times K \) securities returns matrix that satisfies \( \text{rank } D = S \), and let \( \mathbf{q} = \mathbf{p}D \). If \( (\mathbf{p}, (x_{i0}^i, \mathbf{x}^i)_{i \in N}) \) is an Arrow-Debreu equilibrium for the economy \( E = (S, (u^i, (\hat{x}_0^i, \mathbf{\hat{x}}^i)_{i \in N})) \), then there is a profile \( (\mathbf{y}^i)_{i \in N} \) of portfolios for which \( (\mathbf{q}, (\mathbf{y}^i)_{i \in N}, (x_{i0}^i, \mathbf{x}^i)_{i \in N}) \) is an equilibrium of the securities markets defined by \( D \) for the economy \( E \). Conversely, if \( (\mathbf{q}, (\mathbf{y}^i)_{i \in N}, (x_{i0}^i, \mathbf{x}^i)_{i \in N}) \) is a securities-markets equilibrium, then \( (\mathbf{p}, (x_{i0}^i, \mathbf{x}^i)_{i \in N}) \) is an Arrow-Debreu equilibrium for \( E \).

**Remark:** Note that the allocation \( (x_{i0}^i, \mathbf{x}^i)_{i \in N} \) is the same in both equilibria — i.e., everyone’s state-contingent consumption is the same in both equilibria.

**Proof of the Theorem:** This is a simple corollary of the preceding proposition. For each \( i \in N \), we let \( x_{i0}^i - \hat{x}_0^i \) and \( \mathbf{x}^i - \mathbf{\hat{x}}^i \) play the roles of \( z_0 \) and \( \mathbf{z} \) in the proposition. The set \( A \) in the proposition is therefore the set of plans \( (x_{i0}^i, \mathbf{x}^i) \) available to consumer \( i \) — consumer \( i \)’s budget constraint — at the equilibrium price-list \( \mathbf{p} \) in the Arrow-Debreu equilibrium, and the set \( B \) is the set of plans available to him at the securities prices \( \mathbf{q} = \mathbf{p}D \) in the corresponding securities markets. If \( \text{rank } D = S \), then the two sets of available plans \( (x_{i0}^i, \mathbf{x}^i) \) are identical, and the consumer will therefore choose the same plan when facing either price-list. Therefore the utility-maximization and market-clearing conditions are satisfied in one case if and only if they are satisfied in the other case.