FINITE-DIFFERENCE SOLUTION TO THE 2-D HEAT EQUATION

MSE 350
Problem Overview

Given:
- Initial temperature in a 2-D plate
- Boundary conditions along the boundaries of the plate.

Find: Temperature in the plate as a function of time and position.
Energy equation:

\[ \rho C_p \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \]

- \( T(x, 0, t) = \text{given} \)
- \( T(x, H, t) = \text{given} \)
- \( T(0, y, t) = \text{given} \)
- \( T(W, y, t) = \text{given} \)
- \( T(x, y, 0) = \text{given} \)
Approach: *discretize* the temperatures in the plate, and convert the heat equation to finite-difference form.
\[ T^k_{i,j} \]

\[ i, j = \text{location (node numbers)} \]

\[ k = \text{time (time step number)} \]
Discretizing the Heat Equation (Explicit)

\[ \Delta x, \Delta y = \text{node spacings in the } x \text{ and } y \text{ directions.} \]

\[ \frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \]

\[ \frac{T^{k+1}_{i,j} - T^k_{i,j}}{\Delta t} = \alpha \left[ \left( \frac{T^k_{i-1,j} - 2T^k_{i,j} + T^k_{i+1,j}}{\Delta x^2} \right) + \left( \frac{T^k_{i,j-1} - 2T^k_{i,j} + T^k_{i,j+1}}{\Delta y^2} \right) \right] \]
If $\Delta x = \Delta y = h$:

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\frac{T_{i,j}^{k+1} - T_{i,j}^k}{\Delta t} = \alpha \left( \frac{T_{i,j-1}^k + T_{i-1,j}^k - 4T_{i,j}^k + T_{i+1,j}^k + T_{i,j+1}^k}{h^2} \right)$$
Discretizing the Heat Equation (Explicit)

If $\Delta x = \Delta y = h$:

\[
\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
\]

\[
T_{i,j}^{k+1} = T_{i,j}^k + \Delta t \alpha \left( \frac{T_{i,j-1}^k + T_{i-1,j}^k - 4T_{i,j}^k + T_{i+1,j}^k + T_{i,j+1}^k}{h^2} \right)
\]
If $\Delta x = \Delta y = h$:

$$T_{i,j}^{k+1} = \left(1 - \frac{4\Delta t\alpha}{h^2}\right) T_{i,j}^k + \Delta t\alpha \left(\frac{T_{i,j-1}^k + T_{i-1,j}^k + T_{i+1,j}^k + T_{i,j+1}^k}{h^2}\right)$$

Coefficient on $T_{i,j}^k$ must be non-negative for stability.
Hence,

\[
\left( 1 - \frac{4 \Delta t \alpha}{h^2} \right) \geq 0
\]

So

\[
\Delta t \leq \frac{h^2}{4 \alpha}
\]
What do we do about the edges? Same as in a 1-D bar.
What do we do about the edges?

- If we know the temperature of the boundaries already, we don’t need to write equations for those nodes.
- If we know the temperature derivative there, we invent a *phantom node* such that $\frac{\partial T}{\partial x}$ or $\frac{\partial T}{\partial y}$ at the edge is the prescribed value.
At steady-state, time derivatives are zero:

\[
\left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = 0
\]

\[
\left[ \left( \frac{T_{i-1,j}^k - 2T_{i,j}^k + T_{i+1,j}^k}{\Delta x^2} \right) + \left( \frac{T_{i,j-1}^k - 2T_{i,j}^k + T_{i,j+1}^k}{\Delta y^2} \right) \right] = 0
\]
Same $\Delta x$ and $\Delta y (\equiv h)$:

$$T_{i,j} = \frac{1}{4} (T_{i,j-1} + T_{i-1,j} + T_{i+1,j} + T_{i,j+1})$$