Title: Bertrand-Edgeworth Competition, Demand Uncertainty, and Asymmetric Outcomes*

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Abstract: We analyze investment and pricing incentives in a symmetric Bertrand-Edgeworth framework with uncertain demand. Firms choose production capacities before observing demand. Prices are chosen after demand is observed. If the extent of demand variation exceeds a threshold level then a symmetric equilibrium in pure strategies for capacities does not exist. A smaller firm has no incentive (ex ante) to expand its capacity because capacity expansion would reduce its expected revenue in the event that demand is lower than expected. Output prices are predicted to have positive variance when demand is low and zero variance when demand is high.

1. Introduction

In this paper we explore how demand variability and relatively inflexible production capacity interact to influence pricing and investment incentives in an oligopoly setting. Our point of departure is the Kreps and Scheinkman [7] two-stage model of capacity investment and pricing. Kreps and Scheinkman (hereafter, KS) show that in a game in which duopolists choose capacities in stage one and prices in stage two, the equilibrium involves production levels and capacities that correspond to Cournot output levels and prices that are set to clear the market. Their analysis provides a sort of vindication of the Cournot analysis by showing that a simple reformulation exists in which firms actually choose their prices, and yet the final outcome replicates the outcome of the basic Cournot analysis.

We introduce demand uncertainty into the KS formulation. At the time when firms choose their capacities in stage one, the firms are uncertain about the level of demand. After installing their capacities, the firms observe the realization of demand and choose prices in stage two. The timing of decisions and access to demand information that we specify are consistent with the view that capacity choice is a long-run decision made before all market conditions are known and that price choice is a short-run decision made after observing current market conditions.

We show that a symmetric equilibrium in pure strategies for capacity choices fails to exist if the degree of variability in demand exceeds a threshold level, in spite of the fact that firms' payoffs are symmetric. Asymmetric equilibria are shown to exist for particular assumptions on the demand function and the distribution of demand shocks. The capacities in such an asymmetric equilibrium are not equal to Cournot outputs for the corresponding game of output choice with demand uncertainty. Hence, this paper provides another instance in which the KS "vindication" of Cournot is not robust.

The absence of symmetric equilibria is based on a difference in expected revenue between a large (high capacity) firm and a smaller (lower capacity) firm, when demand is low. If demand is low, then the marginal revenue associated with extra capacity for a large firm is zero; the large firm's expected revenue is completely determined by the smaller firm's capacity. However, if demand is low then the marginal revenue associated with extra capacity for a smaller firm may be negative.[1] Additional capacity for the smaller firm lowers the distribution of subgame equilibrium prices in a way that may reduce expected revenue for the smaller firm. This difference in marginal revenue between a large firm and a smaller firm yields a discontinuity in the capacity reaction function at symmetric capacity pairs.

This result reveals an error in a paper that analyzes collusion among firms facing fluctuating demand. Staiger and Wolak [14] formulate a model in which our two-stage game is infinitely repeated. Defections from collusion are punished by reversion to an equilibrium of the two-stage game. Staiger and Wolak identify a symmetric equilibrium in capacity choices for the Nash reversion game; however, their proof fails to recognize the lack of differentiability of the expected revenue function at symmetric capacity pairs. The collusive outcome for the repeated game can not be supported as an equilibrium by the symmetric pure strategies in capacity choices that they identify. It is possible to support a collusive outcome that is qualitatively similar to the one analyzed by Staiger and Wolak in [14] through the use of symmetric optimal punishment strategies of the type considered by Abreu [1].

Another result from our analysis is that the variance of prices may be higher for a low demand realization than for a high demand realization. We find that the type of pricing subgame equilibrium, pure or mixed, depends on the realization of demand. For high demand realizations, the pricing equilibrium is in pure strategies with the firms utilizing all of their capacity. However, for low demand realizations, the subgame pricing equilibrium may be in mixed strategies, with total output less than capacity. The equilibrium in the KS model always involves firms producing at full capacity. We indicate the implications of our results for empirical work in the last section of the paper.

2. The Model and Preliminary Results

We analyze a two-stage duopoly model of capacity investment and pricing in a homogeneous product market. In stage one firms simultaneously invest in production capacity. The firms are uncertain about the size of the market for their output when they make investment decisions. Before stage two, the uncertainty about the size of the market is resolved. The firms engage in price competition in stage two, having observed the level of market demand and investments made in stage one.

The amount of capacity chosen by firm one is x; the amount of capacity chosen by firm two is y. Let a represent the level of market demand. In stage one the firms do not know a but they do know that a will be the realization of a random variable with a particular distribution and with support contained in $[\underline{a},\overline{a}]$. In stage two firms choose their prices after observing production capacities and the level of demand. A subgame in stage two is defined by the triple, (x,y,a).

Our assumptions on the demand function follow KS, with the added feature of a demand level variable. Let the market inverse demand function be $P(q,a)$ (price as a function of quantity q, given demand level a) and let the corresponding ordinary demand function be $D(p,a)$ (quantity demanded as a function of price p, given a).

ASSUMPTION 1. *The function* $P(q,a)$ *is strictly positive for all* $q \in (0,q(a))$*, where* $q(a) \equiv \min\{q \geq 0 : P(q,a) = 0\}$. $P(q,a)$ *is twice continuously differentiable, strictly decreasing, and concave in q for* $q \in (0,q(a))$. *For* $q \geq q(a)$, $P(q,a) = 0$. *For* (q,a) *such that* $P(q,a)$ *is positive,* $P(q,a)$ *is strictly increasing in a.*

We must specify how customers are rationed when the firm offering the low price has insufficient capacity to serve all customers who wish to buy. We follow the specification used by Kreps and Scheinkman [7] and Osborne and Pitchik [9], [10].

ASSUMPTION 2. *Demand is rationed according to the efficient-rationing rule.*[2]

There are no costs of production, apart from capacity investment costs. A constant marginal cost of production for output up to a firm's capacity can be introduced by subtracting this marginal cost from the inverse demand function.

ASSUMPTION 3. *The marginal cost of capacity is constant and equal to c for each firm, where,* $0 < c < E[P(0,a)]$.[3]

While we do not model firms as choosing their output levels directly, it will prove to be useful to have results from Cournot games of output choice at our disposal. Consider firm i's optimal output choice when its rival selects output $q_j \leq q(a)$.

$$b(q_j,a) \equiv \underset{0 \leq q_i \leq q(a)-q_j}{\arg\max} \{q_i(P(q_i+q_j,a))\}$$

$b(q_j, a)$ is the best response function in a Cournot output choice game if *i*'s rival puts q_j on the market, when production is costless and when the demand level is *a*. In Lemma One KS show that the following results follow from Assumptions 1 and 2:

(*i*) *b* is nonincreasing in q_j, is continuously differentiable and strictly decreasing in q_j over the range where it is positive (given *a*),

(*ii*) $b_1(q_j, a) = \dfrac{\partial b(q_j, a)}{\partial q_j} \geq -1$, with strict inequality for (q_j, a) such that $b(q_j, a) > 0$,

(*iii*) there is a unique Nash equilibrium for each demand level *a*, such that each firm chooses output $\hat{q}(a)$, satisfying $\hat{q}(a) = b(\hat{q}(a), a)$.

Define the function, $R(q_j, a) \equiv b(q_j, a) P(b(q_j, a) + q_j, a)$. This function indicates the revenue a firm earns by choosing its best output response to rival output q_j when the level of demand is *a*.

We consider a second Cournot output choice game in which outputs are chosen before the demand level realization and production has constant marginal cost, *c*. The equilibrium of this game serves as a benchmark against which results from the two-stage game may be compared. Consider firm *i*'s optimal output choice when its rival selects output $q_j \leq q(\overline{a})$.

$$b_c(q_j) \equiv \underset{0 \leq q_i \leq q(\overline{a}) - q_j}{\arg\max} \{E[q_i P(q_i + q_j, a) - cq_i]\}$$

Under assumptions 1–3 the objective function is twice continuously differentiable and strictly concave in q_i. Thus, there is a unique best response for each $q_j \in (0, q(\overline{a}))$. The condition $c < E[P(0, a)]$ from assumption 3 implies that $b_c(0) > 0$. The properties of b_c are analogous to those of *b*, as specified in the preceding paragraph. The reaction function

b_c is nonincreasing in q_j and is continuously differentiable with $0 > b_c'(q_j) > -1$ over the range where it is positive. Therefore, there is a unique Nash equilibrium for this second output choice game such that each firm chooses output $\tilde{q}_c$, satisfying $\tilde{q}_c = b_c(\tilde{q}_c)$.

We now return to our analysis of the two-stage model of investment and pricing. A pricing subgame is defined by the triple, (x,y,a). The nature of subgame equilibria can be described by dividing the capacity space into three regions, as depicted in Figure One:

$$A(a) = \{(x,y)\colon x \geq 0,\ y \geq 0,\ x \leq b(y,a),\ y \leq b(x,a)\},$$

$$B(a) = \{(x,y)\colon x \geq q(a), y \geq q(a)\}, \text{ and}$$

$$C(a) = \{(x,y)\colon x \geq 0, y \geq 0, (x,y) \notin A(a), (x,y) \notin B(a)\}.$$

In region $A(a)$ the subgame equilibrium is in pure strategies, with each firm setting a price equal to the market clearing price. The market clearing price is determined by the intersection of demand with a vertical supply at quantity $x + y$. In region $B(a)$ the subgame equilibrium is a Bertrand equilibrium with price equal to zero (marginal production cost) for both firms. If capacities are in region $C(a)$ then there is no pure strategy subgame Nash equilibrium in prices (except for the trivial case in which one firm has zero capacity); there is a unique mixed strategy equilibrium in prices ([10, Theorem 1]).

Our analysis focuses on the expected revenue in subgame equilibria rather than on the form of subgame equilibrium strategies. The following lemma summarizes results from [7, Lemma 5] and [9, Theorem, Table 1].

LEMMA *Expected revenue for firm one in a subgame equilibrium is the function* $r(x, y, a)$ *, which satisfies the following conditions:*

$$
\begin{aligned}
&r(x,y,a) = P(x+y,a)x && ,if\ (x,y) \in A(a) \\
&r(x,y,a) = \frac{x}{y}R(x,a) && ,if\ (x,y) \in C(a), x \le y, and\ R(x,a) \le yP(y,a) \\
&r(x,y,a) = \tilde{s}x && ,if\ (x,y) \in C(a), x \le y, and\ R(x,a) > yP(y,a), \\
& && where\ \tilde{s}\ satisfies, \tilde{s}D(\tilde{s},a) = R(x,a) \\
&r(x,y,a) = R(y,a) && ,if\ (x,y) \in C(a), x > y \\
&r(x,y,a) = 0 && ,if\ (x,y) \in B(a)
\end{aligned}
$$

Expected revenue for firm 2 is given by $r(y, x, a)$. *The function* $r(\ )$ *is continuous in x and y for each value of a.*

In region $C(a)$ the expected revenue of the large firm depends on the level of demand and the capacity of the smaller firm. The large firm's expected revenue is independent of its own capacity. The small firm's expected revenue in region $C(a)$ takes one of two forms. For capacities in what we refer to as Region I, expected revenue for a small firm is the fraction y/x of the larger firm's expected revenue (where y is the small firm's capacity). For capacities in Region II, the small firm's expected revenue is $\tilde{s}y$, where $\tilde{s}$ is an implicit function of the small firm's capacity and the level of demand.

3. Main Results

The first theorem concerns existence of symmetric equilibria for capacity choices when there is a continuous distribution of demand levels. A corollary provides an easily interpretable version of the result for the special case of additive demand shocks. The second theorem characterizes equilibrium capacity choices for a linear demand function and a two-point distribution of demand levels.

THEOREM 1. *Suppose that the probability density for the demand shift variable is positive and continuous on its support,* $[\underline{a}, \overline{a}]$, *where* $0 \le \underline{a} < \overline{a} < \infty$.

(i) *If* $\hat{q}(\underline{a}) \ge \tilde{q}_c$ *then there is a symmetric equilibrium in capacity choices, with capacity equal to* $\tilde{q}_c$ *for each firm. Capacity is fully utilized and prices are set equal to the market clearing level for every demand realization in this equilibrium.*

(ii) *If* $\hat{q}(\underline{a}) < \tilde{q}_c$ *then a symmetric equilibrium in pure strategies for capacity choices does not exist.*

Proof: See the appendix.

Part (i) is similar to the well-known result of Kreps and Scheinkman [7]. They show that the equilibrium for a game with certain demand involves prices that are set to clear the market, fully utilized capacity, and capacity levels that correspond to Cournot output levels. Part (i) applies to situations in which the Cournot output level for the game with no costs and the lowest possible demand level $(\hat{q}(\underline{a}))$ is greater than or equal to the Cournot output level for the game with marginal cost $c > 0$ and uncertain demand $(\tilde{q}_c)$. The payoffs for capacity investment are then essentially equivalent to the payoffs associated with the game of output choice in which output is chosen prior to the realization of the demand level and firms have marginal cost c. The symmetric equilibrium capacities correspond to Cournot duopoly outputs for such a game of output choice. Capacities are fully utilized and prices are set to clear the market for every demand realization in equilibrium.

Part (*ii*) stems from the properties of the expected revenue function for capacities in region *C*(*a*). If $\hat{q}(\underline{a})$ is less than $\tilde{q}_c$ then any viable candidate for a symmetric equilibrium will be in a subgame in region *C*(*a*) with positive probability. In region *C*(*a*) there is an asymmetry in the effect of a change in capacity on a firm's expected revenue for a subgame. A change in capacity for a large firm has no effect on its expected revenue. The expected revenue for a small firm declines for an increase in its capacity, when its capacity is nearly as large as its rival's capacity (capacities are in Region I of *C*(*a*)). This asymmetry in expected marginal revenue due to a capacity change leads to a "kink" in a firm's expected profit function at symmetric capacity pairs, as long as there is a positive probability that capacities are in region *C*(*a*), for some realizations of *a*.

Part (*ii*) of Theorem One rules out existence of symmetric equilibria in pure strategies for capacity choices. Lemma Seven of Dasgupta and Maskin [2] may be applied to prove existence of a symmetric mixed strategy equilibrium in capacity choices.

The Corollary shows that the hypotheses of Theorem One have a simple interpretation for the case of additive demand shocks. The existence of a symmetric equilibrium in pure strategies for capacities turns on the extent to which demand may deviate below its mean value.

COROLLARY. *Suppose that the demand shock is additive, so that inverse demand may be expressed as,* $P(q,a) = \breve{P}(q) + a$ *, for* $q \le q(a)$*. If* $\underline{a} \ge E(a) - c$ *then there is a symmetric equilibrium in pure strategies for capacity choices, in which capacity is fully utilized and prices are set equal to the market clearing level for every demand realization. If* $\underline{a} < E(a) - c$ *then a symmetric equilibrium in pure strategies for capacity choices does not exist.*

The Corollary is proved by translating the comparisons of $\hat{q}(\underline{a})$ and $\tilde{q}_c$ in Theorem One into comparisons of $\underline{a}$ and $E(a)-c$.[4]

THEOREM 2. *Suppose that the demand level is* $\underline{a} > 0$ *with probability* $\theta \in (0,1)$ and $\overline{a} > \underline{a}$ *with probability* $1-\theta$, *and that inverse demand is* $P(q,a) = a - q$, *for* $q \leq a$, *and* $P(q,a) = 0$, *for* $q > a$.

(i) *If* $\underline{a} \geq E(a) - c$ *then there is a symmetric equilibrium in capacity choices, with capacity equal to* $\tilde{q}_c = (E(a)-c)/3$ *for each firm. Capacity is fully utilized and prices are set equal to the market clearing level for every demand realization in this equilibrium.*

(ii) *If* $\frac{1}{2}\overline{a} \leq \underline{a} < E(a) - c$ *then a symmetric equilibrium in pure strategies for capacity choices does not exist; there are pure strategy asymmetric equilibria in capacity investments. In the low demand subgame firms play asymmetric mixed pricing strategies and set prices above the market clearing level. Prices are set at the market clearing level and capacity is fully utilized in the high demand subgame.*

Proof: See the appendix.

Part (i) is quite similar to part (i) of Theorem One and the Corollary. The only difference is that uniqueness of equilibrium capacities is established for the model with a two-point demand distribution.

The result in part (ii) is illustrated in Figure Two, which provides a diagram of capacity reaction functions, for $x > y$. The reaction function for firm two, $\tilde{R}^2$, has a downward jump across the 45 degree line because of the discontinuity in marginal revenue of capacity (when demand is low) at symmetric capacities and another downward jump across the boundary of Regions I and II. The condition, $\overline{a} < 2\underline{a}$, insures that $\tilde{R}^2$ crosses the reaction curve, $\tilde{R}^1$, of the large firm in Region I.

What sustains the asymmetric position of firms as an equilibrium outcome? Note that in the high demand state the marginal revenue for capacity for the larger firm is *smaller* than the marginal revenue for capacity for the smaller firm. Since the firms have the same marginal cost of capacity investment, there must be a corresponding difference in the marginal revenue for capacity in the low demand state. Specifically, marginal revenue for capacity for the smaller firm in the low demand state must be *negative*, since marginal revenue for capacity for the larger firm is zero in the low demand state.

When demand is low, additional capacity for the small firm lowers the distribution of subgame equilibrium prices in a way that reduces expected revenue for the large firm and may reduce expected revenue for the small firm. Recall from the Lemma that the small firm's revenue in Region I is a fraction of the large firm's revenue; firm two's expected revenue is $\frac{y}{x}R(y,\underline{a})$ for $y < x$. Given the assumptions of Theorem Two, as the small firm expands its capacity y beyond its equilibrium capacity, y*, the revenue function $R(y,\underline{a})$ decreases faster than the ratio of capacities y/x^* increases, and so the small firm's marginal revenue is *negative* in the low demand state. Thus, it is the possibility that demand will be low that deters the small firm from expanding (in equilibrium) because extra capacity will actually *lower* its (expected) revenue.

The other force that maintains the asymmetric equilibrium outcome may be traced to the fact that the expected marginal revenue of capacity for the large firm is *zero* in the low demand state. If demand turns out to be low then a large firm would have gained nothing by having less capacity, provided its capacity remains above its rival's capacity. At an asymmetric equilibrium, the expected marginal revenue of capacity due to the high demand state is balanced by the marginal cost of capacity. If the large firm reduces its capacity below its equilibrium level then the expected revenue loss will exceed the capacity cost savings.

A characterization of pure strategy equilibria for general demand functions and demand shock distributions is a challenging problem. As our analysis has demonstrated, payoff functions are not differentiable everywhere and best response correspondences may have "jumps". It is possible that no pure strategy equilibria exist for some demand and distribution configurations.

One possible approach for a more general analysis would be to apply the theory of submodular games, as in Topkis [15]. This approach requires that expected payoffs exhibit non-increasing differences in capacities[5]; differentiability is not required. Pure strategy equilibria exist for submodular games, even though best response correspondences may have jumps. However, this approach cannot be applied directly to our model, since payoffs do not satisfy non-increasing differences for all capacity pairs. For example, if capacities are in Region I (see Figure One) then expected subgame revenue for firm two in the low demand subgame can have a positive cross partial derivative in capacities. This can yield a positive cross partial derivative for total expected payoff for firm two, which would be inconsistent with the submodular approach. There may be restrictions on parameters or functional forms that would allow one to apply the submodular games approach.

4. Related Literature and Discussion

The model analyzed by Gabszewicz and Poddar [4] is similar to ours except that firms choose outputs rather than prices in the final stage subgame. They prove that if firms fully utilize capacity in all demand states in equilibrium then equilibrium capacities correspond to certainty equivalent Cournot capacities (in terms of our notation, each firm chooses capacity, $\tilde{q}_c$). This is analogous to our result in part (*i*) of Theorem One.

We believe that our formulation has the advantages of providing an explicit model of price formation and of generating richer implications for the impact of demand fluctuations on pricing compared to the model of Gabszewicz and Poddar. Our analysis predicts that the variance of output prices in industries with capacity-constrained, price-setting firms is higher during low demand periods (due to mixed strategy pricing) than in high demand periods. Wilson and Reynolds [16] examine this prediction using observations of industrial output and prices over the business cycle. They find that the estimated variance of prices is significantly higher when demand is low than when demand is high in almost all U.S. manufacturing industries.

Several related oligopoly analyses have formulated output or capacity choice models that yield asymmetric outcomes. Saloner [12] analyzes a Cournot duopoly model in which firms may spread production over two periods. The opportunity to observe and react to a rival's initial output choice induces multiple subgame perfect Nash equilibria (SPNE), including all outcomes on the outer envelope of the best-response functions between the firms' Stackelberg outcomes. Pal [11] generalizes Saloner's model by allowing production costs to differ over time (say, due to time discounting). If costs fall slightly over time then asymmetric equilibria remain. Otherwise, the model generates a unique, symmetric SPNE. More recently, Maggi [8] utilizes the sequential decision structure of [11] and [12] to analyze the role of sequential capacity investment in generating asymmetric duopoly outcomes. Maggi introduces demand uncertainty into the model, with the realization of demand revealed between the first and second investment stages. The nature of outcomes depends on the demand level. If demand is low then asymmetric outcomes emerge, but if demand is high then firms invest up to a symmetric outcome.

Our analysis is similar to Maggi [8] in that we analyze the interaction of capacity investment and demand uncertainty in generating asymmetric outcomes. We also use a sequential decision structure. However, we assume that a single stage of capacity investment is followed by a single stage of pricing, as in Kreps and Scheinkman [7], rather than utilizing a sequential capacity or output choice formulation. Asymmetric outcomes are generated in our model by the possibility of low demand realizations and the corresponding implications for the marginal revenue of capacity.

A related result appears in Davidson and Deneckere [3]. They modify the Kreps and Scheinkman analysis to allow random demand rationing, rather than efficient rationing. This change in the rationing rule can create incentives for capacity investment such that excess capacity emerges in equilibrium. For sufficiently small costs of capacity, Davidson and Deneckere find that reaction curves (in capacity space) are discontinuous and that only asymmetric equilibria exist.

We view our model as a complement to the sequential output/capacity approach in [8], [11] and [12] and to the random rationing approach in [3], rather than as a substitute for these formulations. We conjecture that the inclusion of sequential capacity choice and random demand rationing in our formulation would expand the range of parameters over which asymmetric outcomes emerge and perhaps increase the magnitude of asymmetries.

Staiger and Wolak [14] formulate and analyze a model that is a special case of the Corollary (with, $P(q,a) = a - q$ and $\underline{a} = 0 < E(a) - c$). Their analysis of the two-stage game is then used to characterize collusion when the game is infinitely repeated. Staiger and Wolak identify a symmetric capacity outcome for our two-stage model as the equilibrium to which the firms revert following a deviation from the collusive path in the repeated game. Their proof, however, fails to recognize the lack of differentiability of the revenue function at symmetric capacity pairs. In terms of our notation, they specify the derivative of $r(x, y, a)$ with respect to x at a symmetric capacity pair to be zero in region $C(a)$. This is correct for the right hand derivative, but not for the left-hand derivative. As a consequence, Staiger and Wolak incorrectly identify a symmetric outcome in capacities as an equilibrium of the two-stage game.[6]

Why does this difficulty matter for the analysis of collusion? Staiger and Wolak find symmetric collusive capacities and equal collusive profits for the two firms based on the symmetric reversionary outcome that they identify. These collusive capacities are not supported as part of a noncooperative equilibrium if the reversionary outcome is not itself an equilibrium for the stage game. There would appear to be several ways to remedy this difficulty. One approach, which is in fact suggested in Staiger and Wolak [14], would be to use symmetric optimal punishment strategies of the type considered by Abreu [1]. This approach would yield results on optimal collusion that are qualitatively similar to the results reported in [14].[7] Another approach would be to utilize a symmetric mixed strategy equilibrium in capacity choices, which does exist, as noted earlier. This would yield equal expected profits for the two firms in the reversionary equilibrium and could support symmetric collusive capacities (though not the collusive capacities identified by Staiger and Wolak).[8] A third approach would be to characterize an asymmetric equilibrium for the two-stage game, as we do in Part (*ii*) of Theorem Two, and use this as the basis for an asymmetric collusive mechanism. Asymmetric collusion of this type may be an interesting topic for further research.

Notes

1. We show that marginal revenue is negative for a small firm if its capacity is sufficiently close to the larger firm's capacity.

2. See Lemma Six in KS for the specific form of a firm's demand as a function of firms' prices. This assumption is not innocuous. The rationing rule is related to the issue of whether symmetric equilibria exist in the model without demand uncertainty, as pointed out in Davidson and Deneckere [3].

3. This is more restrictive than KS, who allow capacity cost to be an increasing, convex function of capacity.

4. A related result is derived by Hviid [6] in an analysis of our two stage model with linear demand and zero capacity costs. He shows that the equilibrium involves pure strategies only if there is no demand uncertainty. In this case equilibrium capacities equal the Cournot output, $\tilde{q}_c$. This result is a special case of our Corollary. If $c = 0$ then the condition $\underline{a} \geq E(a) - c$ is satisfied only if the support of the demand shock distribution is a single point.

5. If the payoff function is twice differentiable then non-increasing differences corresponds to a non-positive cross partial derivative with respect to the two capacities.

6. A symmetric equilibrium in capacity choices can exist if the lower bound for demand levels is positive and the variance of the distribution of demand levels is small. This case is described in part (*i*) of Theorem One and its Corollary. However, in such an equilibrium, excess capacity never arises; firms set market clearing prices in each possible subgame.

7. In their working paper [13], Staiger and Wolak derive results based on symmetric optimal punishment strategies.

8. Some would argue that reliance on mixed strategies, whether for capacities or prices, is a symptom of model mis-specification. We argue that mixed strategies are easier to justify for prices than for capacities. For example, if there is ex ante uncertainty about short run marginal production cost and firms have private information about their own costs, then an equilibrium may exist in which firms utilize pure strategies for prices, but for which prices exhibit substantial variability. Holt [5] shows that equilibrium pure strategies for prices for a game with cost uncertainty converge to equilibrium mixed strategies as the amount of cost uncertainty vanishes. It is difficult to think of a corresponding interpretation for mixed strategies for capacity choices.

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APPENDIX A: PROOF OF THEOREM 1

Part (*i*). Consider $(x', y') = (\tilde{q}_c, \tilde{q}_c)$ as a candidate for equilibrium capacities. Note that $(x', y') \in A(a)$ for all $a \in [\underline{a}, \overline{a}]$ since $\hat{q}(a) > \hat{q}(\underline{a}) \geq x' = y'$ for $a > \underline{a}$. Is x' a best response for firm one to y'? If $x < x'$ then $(x, y') \in A(a)$ for all $a \in [\underline{a}, \overline{a}]$. By the Lemma firm one's expected profit is, $\pi(x, y') = E[P(x + y', a)x - cx]$. This function is strictly increasing in *x* for $x < x' = \tilde{q}_c$ since the RHS is strictly concave in *x*, and $\tilde{q}_c$ is the optimal output response to $y' = \tilde{q}_c$. If $x > x'$ then there are three cases to consider.

Case 1. ($x < b(y', \underline{a})$): In this case $(x, y') \in A(a)$ for all $a \in [\underline{a}, \overline{a}]$. The marginal payoff for capacity for firm one is, $E[P(x + y', a) + xP_1(x + y', a)] - c$. This marginal payoff is negative since it is the derivative of, $E[P(x + y', a)x - cx]$, which is maximized at $x' = \tilde{q}_c < x$.

Case 2. ($b(y', \underline{a}) \leq x < b(y', \overline{a})$): In this case (x, y') is in $C(a)$ for some *a*-values and in $A(a)$ for other *a*-values. Let $a'(x, y')$ be the value of *a* that satisfies $x = b(y', a'(x, y'))$. Then $(x, y') \in C(a)$ for $a < a'(x, y')$ and $(x, y') \in A(a)$ for $a \geq a'(x, y')$. By the Lemma the marginal payoff for capacity for firm one is,

$$\pi_1(x, y') = \int_{a'(x,y')}^{\overline{a}} [P(x + y', a) + xP_1(x + y', a)]dF(a) - c. \tag{1}$$

If $x = b(y', \underline{a})$ then this marginal payoff is, $\int_{\underline{a}}^{\overline{a}} [P(x + y', a) + xP_1(x + y', a)]dF(a) - c$, which is negative since it corresponds to the marginal payoff of output in a Cournot game with output in excess of the best response. The derivative of the marginal payoff in (1) with respect to *x* is,

$$\pi_{11}(x,y') = \int_{a'(x,y')}^{\bar{a}} [2P_1(x+y',a) + xP_{11}(x+y',a)]dF(a)$$
$$+ \{P(x+y',a'(x,y')) + xP_1(x+y',a'(x,y'))\}\frac{\partial a'(x,y')}{\partial x}$$

The integrand is negative since P is decreasing and concave in output and the term in curly brackets is zero. So, the marginal payoff for capacity for firm one is negative in case two, since $\pi_1(x,y')$ is negative at $x = b(y',\underline{a})$ and $\pi_1(x,y')$ is decreasing in x for $b(y',\underline{a}) \leq x < b(y',\bar{a})$.

Case 3. ($x \geq b(y',\bar{a})$): In this case $(x,y') \in C(a)$ for all $a \in [\underline{a},\bar{a}]$ and $\pi_1(x,y') = -c < 0$.

This demonstrates that firm one's expected profit is increasing in x for $x < x'$ and decreasing in x for $x > x'$, so x' is a best response to y'. By identical arguments, y' is a best response to x'.

Part (*ii*). Let (x',y') be any pair of capacities such that $x' = y'$. Consider (x',y') as a candidate for equilibrium capacities.

If $x' = y' \leq \hat{q}(\underline{a})$ then (x',y') is in $A(a)$ for all a and, by the Lemma, the expected payoff for firm one is, $E[P(x'+y',a)x' - cx']$. The hypothesis of part (*ii*), $\hat{q}(\underline{a}) < \tilde{q}_c$, implies that $x' = y' < \tilde{q}_c$, so that the marginal payoff for capacity for firm one is strictly positive; x' cannot be a best response to y'.

If $x' = y' \geq q(\bar{a})$ then (x',y') is in $B(a)$ for all a and, by the Lemma, the expected revenue is zero for every possible demand realization, and firm one earns payoff, $-cx'$. Any capacity $x < x'$ would yield higher expected payoff for firm one.

If $\hat{q}(\underline{a}) < x' = y' < q(\overline{a})$ then there is a set of *a*-values such that (x', y') is in *C*(*a*). This set is defined as, $\Gamma \equiv \{a \in [\underline{a}, \overline{a}]: \underline{a} \leq a \leq \min\{a'(x'), \overline{a}\}\}$, where $a'(x')$ satisfies, $x' = b(x', a'(x'))$. Consider the marginal revenue for capacity for firm one for any $a \in \Gamma$. By the Lemma the right hand partial derivative of the revenue function is zero since $r(x, y', a)$ is independent of *x* for $x > y'$. That is, $\left.\frac{\partial r(x', y', a)}{\partial x}\right|_{+} = 0$.

Recall that $R(x, a) \equiv b(x, a)P(b(x, a) + x, a)$. R is continuous in *x* given *a*, since the inverse demand function and the best response function, $b(x, a)$, are continuous in *x* given *a*. The aim of this section is to show that $R(x, a) < y'P(y', a)$ for *x* smaller than, but sufficiently close to, $x' = y'$. Note that if $a \in \Gamma$ then $x' > \hat{q}(a)$. This inequality implies that $b(x', a) < \hat{q}(a) < x' = y'$, since under Assumption One, the best response to an output x' is smaller than the Cournot output (x' is greater than the Cournot output). So, given x' and *a*, there exists $\tilde{\varepsilon} \in (0, x' - \hat{q}(a))$ such that $b(x', a) + \tilde{\varepsilon} < x' = y'$. Also, $P(b(x', a) + x', a) \leq P(x', a)$ since inverse demand is non-increasing in output. Combining the two previous inequalities yields,

$$P(b(x', a) + x', a)[b(x', a) + \tilde{\varepsilon}] = R(x', a) + P(b(x', a) + x', a)\tilde{\varepsilon} < x'P(x', a) = y'P(y', a).$$

Given the continuity of R in capacity, for $\varepsilon \equiv \tilde{\varepsilon}P(b(x', a) + x', a) > 0$ there exists $\delta > 0$ such that $R(x, a) < R(x', a) + \varepsilon$, for $x \in (x' - \delta, x')$. Thus, for *x* sufficiently close to $x' = y'$, we have $R(x, a) \equiv b(x, a)P(b(x, a) + x, a) < y'P(y', a)$.

By the Lemma, for *x* smaller than and sufficiently close to $x' = y'$ the expected revenue function for firm one is,

$$r(x, y', a) = \frac{x}{y'} b(x, a) P(b(x, a) + x, a).$$

Firm one's marginal revenue for capacity is,

$$r_1(x, y', a) = \frac{1}{y'}[bP + xb_1 P + xbP_1(b_1 + 1)],$$

where the arguments in functions on the right-hand-side have been dropped in order to express the result more compactly. Marginal revenue simplifies to,

$$r_1(x, y', a) = \frac{1}{y'}[bP + xbP_1 + xb_1(P + bP_1)].$$

The expression, $P + bP_1 = P(b(x,a) + x, a) + b(x,a)P_1(b(x,a) + x, a)$, is zero since $b(x,a)$ is a revenue maximizing output response to *x*, given *a*. The left-hand derivative is,

$$\left.\frac{\partial r(x', y', a)}{\partial x}\right|_{-} = \lim_{x \uparrow x'} r_1(x, y', a) = \frac{b(x',a)}{y'}[P(b(x',a) + x', a) + x'P_1(b(x',a) + x', a)]$$
$$< \frac{b(x',a)}{y'}[P(b(x',a) + x', a) + b(x',a)P_1(b(x',a) + x', a)] = 0$$

The final term in brackets is zero since $b(x',a)$ is the revenue maximizing output response to x' given *a*. The inequality follows because P_1 is negative and $b(x',a) < x'$.

The set Γ has positive measure and the density function for demand shocks is positive and continuous. Therefore, the non-differentiability property of the revenue function carries over to the expected profit function.

$$\left.\frac{\partial \pi(x', y')}{\partial x}\right|_{-} = E\left[\left.\frac{\partial r(x', y', a)}{\partial x}\right|_{-} - c\right] < E\left[\left.\frac{\partial r(x', y', a)}{\partial x}\right|_{+} - c\right] = \left.\frac{\partial \pi(x', y')}{\partial x}\right|_{+}$$

Capacity x' cannot be a best response for firm one to y' when this condition holds.

These arguments show that no symmetric pair of capacities satisfies the mutual best response property required for equilibrium, when $\hat{q}(\underline{a}) < \tilde{q}_c$. ■

APPENDIX B: PROOF OF THEOREM 2

Part (*i*). Consider $y' \in [0, \frac{1}{3}\underline{a}]$. If $x \in [0, b(y', \underline{a})]$ then $(x, y') \in A(\underline{a})$, $(x, y') \in A(\overline{a})$, and

$$\pi(x, y') = (1-\theta)x(\overline{a} - x - y') + \theta x(\underline{a} - x - y') - cx\,.$$

$\pi(x, y')$ is strictly concave for $x \in [0, b(y', \underline{a})]$ and reaches a local maximum at,

$$b_c(y') = \tfrac{1}{2}(\theta\underline{a} + (1-\theta)\overline{a}) - c - y') = \tfrac{1}{2}(E(a) - c - y')\,.$$

Note that $(b_c(y'), y') \in A(\underline{a})$ since $b_c(y') = \frac{1}{2}(E(a) - c - y') \le \frac{1}{2}(\underline{a} - y') = b(y', \underline{a})$ by the hypothesis of the Theorem. Output $b_c(y')$ is also a unique global maximum of $\pi(x, y')$ since π is continuous in x and,

$$\frac{\partial\, \pi(x, y')}{\partial\, x} = (1-\theta)(\overline{a} - 2x - y') - c < 0\,, \text{ for } b(y', \underline{a}) < x < b(y', \overline{a})$$

and

$$\frac{\partial\, \pi(x, y')}{\partial x} = -c < 0\,, \text{ for } x > b(y', \overline{a})\,.$$

Let $\tilde{q}_c = (E(a) - c)/3$ be the proposed equilibrium capacity per firm. Note that $\underline{a} \ge E(a) - c$ implies that $\tilde{q}_c \le \frac{1}{3}$ and that $b_c(\tilde{q}_c) = \tilde{q}_c$. So, $(\tilde{q}_c, \tilde{q}_c)$ is an equilibrium and is the only equilibrium in which both outputs are less than or equal to $\frac{1}{3}\underline{a}$. Uniqueness may be established by ruling out equilibria in which one or both outputs exceed $\frac{1}{3}\underline{a}$.

Part (*ii*). If $\frac{1}{2}\overline{a} \le \underline{a} < E(a) - c$ then there is no equilibrium involving symmetric capacities. The proof of this result is analogous to the proof of part (*ii*) of Theorem One. The restriction that $\frac{1}{2}\overline{a} \le \underline{a}$ is sufficient to rule out symmetric equilibria in which capacities are in the set $B(\underline{a})$ (where both firms earn zero revenue when demand is low).

We seek a characterization of an equilibrium with $x > y$. We will suppose that equilibrium capacities are in Region I of $C(\underline{a})$ and then verify that this is the case given the hypothesized parameter restrictions. If demand is low then expected revenue in the pricing subgame for firm two in Region I is,

$$r(y,x,\underline{a}) = \frac{y}{x}R(y,\underline{a}) = \frac{y}{4x}(\underline{a}-y)^2 .$$

If demand is low then expected revenue in the pricing subgame for firm one is,

$$r(x,y,\underline{a}) = R(y,\underline{a}) = \tfrac{1}{4}(\underline{a}-y)^2 .$$

Expected profit for firm one in stage one is,

$$\pi(x,y) = \theta R(y,\underline{a}) + (1-\theta)x(\overline{a}-x-y) - cx .$$

Optimal capacity choice for firm one satisfies,

$$\pi_1(x,y) = (1-\theta)(\overline{a}-2x-y) - c = 0$$

or,

$$x = \tfrac{1}{2}(\overline{a} - \frac{c}{1-\theta} - y) .$$

Expected profit for firm two in stage one is,

$$\pi(y,x) = \theta \frac{y}{x} R(y,\underline{a}) + (1-\theta)y(\overline{a}-x-y) - cy .$$

Optimal capacity choice for firm two satisfies,

$$\pi_1(y,x) = \frac{\theta}{4x}(\underline{a}-y)(\underline{a}-3y) + (1-\theta)(\overline{a}-x-2y) - c = 0 .$$

The following notation will prove to be convenient. Let $\psi \equiv \frac{\overline{a}}{\underline{a}} - \frac{c}{(1-\theta)\underline{a}}$. The condition $\underline{a} < E(a) - c$ is equivalent to $\psi > 1$. Also, let $X = \frac{x}{\underline{a}}$ and $Y = \frac{y}{\underline{a}}$. The optimal capacity choice condition for firm one may now be expressed as,

$$X = \tfrac{1}{2}(\psi - Y), \tag{2}$$

and the optimal capacity choice for firm two may be expressed as,

$$\frac{\theta}{4X}(1-Y)(1-3Y) + (1-\theta)(\psi - X - 2Y) = 0. \tag{3}$$

Substituting for *X* from (2) into (3) yields,

$$\theta(1-Y)(1-3Y) + (1-\theta)(\psi - Y)(\psi - 3Y) = 0. \tag{4}$$

The LHS of (4) is a weighted sum of two quadratics in *Y*. One solution lies between 1/3 and ψ/3, the smaller roots of the two quadratics; this solution is the economically relevant one. Let *Y** be the solution in the interval, (1/3, ψ/3). Then, $X^* = \tfrac{1}{2}(\psi - Y^*) > \psi/3 > Y^*$, as required for the payoff functions used in these derivations. Let $x^* = \underline{a}X^*$ and $y^* = \underline{a}Y^*$.

The next step is to verify that (*x**, *y**) is in Region I for firm two when demand is low. The boundary between Regions I and II in $C(\underline{a})$ satisfies, $R(y, \underline{a}) = xP(x, \underline{a})$ or, $y = \beta(x) \equiv \underline{a} - 2\sqrt{x(\underline{a} - x)}$. A sufficient condition for (*x**, *y**) to be in Region I is that the best response for firm one to $y = \underline{a}/3$ be smaller than the value of *x* satisfying, $\beta(x) = \underline{a}/3$. In terms of Figure Two, this condition insures that firm one's reaction function, $\tilde{R}^1$, is to the left of the boundary between Regions I and II, for $y \in \left(\frac{\underline{a}}{3}, \frac{\underline{a}\psi}{3}\right)$.

Firm one's best response to $y = \underline{a}/3$ is $x = \frac{1}{2}(\underline{a}\psi - \underline{a}/3) = (3\psi - 1)\underline{a}/6$. The condition $\bar{a} < 2\underline{a}$ implies that $\psi < 2$, which implies that $\beta^{-1}(\underline{a}/3) = (3+\sqrt{5})\underline{a}/6 > (3\psi - 1)\underline{a}/6$. So, the hypothesis of part (*ii*) implies that (x*, y*) is in Region I.

The preceding arguments show that y* is a best response to x* when y is restricted to values in Region I. This leaves open the possibility that the best response to x* is in Region II. In terms of Figure Two, $\tilde{R}^2$ may have a downward jump discontinuity before it crosses $\tilde{R}^1$ in Region I. In Region II, expected revenue for firm two in the low demand subgame is independent of x;

$$r(y, x^*, \underline{a}) = \tfrac{1}{2} y[\,\underline{a} - \sqrt{y(2\underline{a} - y)}\,].$$

This function is maximized at, $y_{II} = \underline{a}\left(\frac{2-\sqrt{3}}{2}\right)$, and

$$r(y_{II}, x^*, \underline{a}) = \underline{a}^2\left(\frac{2-\sqrt{3}}{8}\right).$$

The capacity pair $(x^*, \underline{a}/3)$ is in Region I if demand is low. In this case, firm two's expected revenue in the subgame is,

$$r(\underline{a}/3, x^*, \underline{a}) = \frac{\frac{1}{3}\underline{a}}{4x^*}(\underline{a} - \tfrac{1}{3}\underline{a})^2 = \frac{\underline{a}^3}{27x^*},$$

which exceeds $r(y_{II}, x^*, \underline{a})$, since $x^* < \underline{a}$ (the result that (x*, y*) is in Region I implies that $x^* < \underline{a}$). If demand is high then firm two has positive marginal revenue for capacity $y \in (0, y^*)$, given x* (since $(x^*, y) \in A(\bar{a})$). So,

$$\pi(\underline{a}/3, x^*) = \theta r(\underline{a}/3, x^*, \underline{a}) + (1-\theta)(\bar{a} - x^* - \underline{a}/3)\underline{a}/3 - c\underline{a}/3$$

$$> \theta r(y_{II}, x^*, \underline{a}) + (1-\theta)(\bar{a} - x^* - \beta(x^*))\beta(x^*) - c\beta(x^*) \geq \max_{y \in [0, \beta(x^*)]} \pi(y, x^*)$$

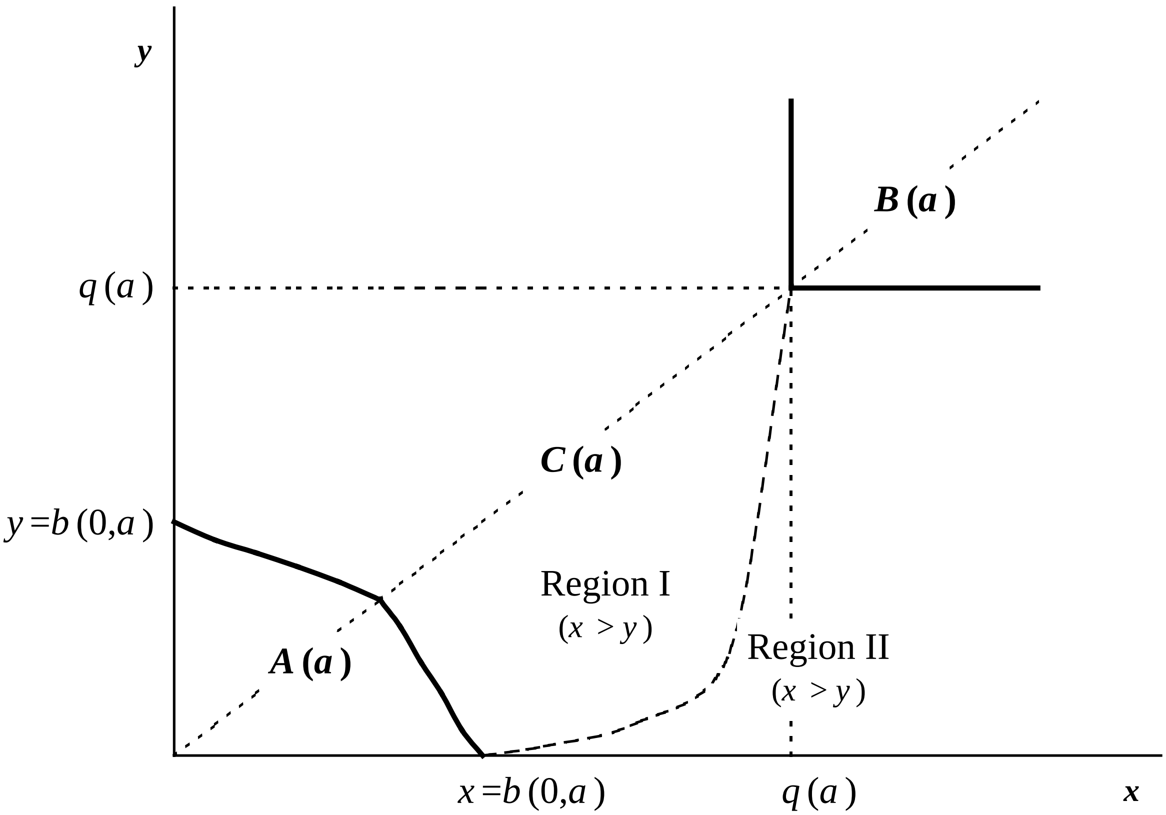
where the strict inequality follows because $\beta(x^*) < \underline{a}/3$ and the small firm's marginal revenue for capacity is positive if demand is high. The capacity pair $(x^*, \underline{a}/3)$ is in Region I for firm two and $y = y^*$ is the best response to x^* for firm two within Region I.

Therefore, $\pi(y^*, x^*) \geq \pi(\underline{a}/3, x^*) > \max_{y \in [0, \beta(x^*)]} \pi(y, x^*)$.

Given that firm one chooses capacity x^*, all capacity choices for firm two in Region II of $C(\underline{a})$ yield lower expected profit for firm two than y^*. ■

Figure One

Capacity Regions for the Pricing Subgame



$A(a) = \{ (x, y): x \geq 0, y \geq 0, x \leq b(y, a), \; y \leq b(x, a)\}$
$B(a) = \{ (x, y): x \geq q(a), y \geq q(a)\}$
$C(a) = \{ (x, y): x \geq 0, y \geq 0, (x, y) \notin A(a), \; (x, y) \notin B(a)\}$

Figure Two

Capacity Reaction Curves ($x > y$)