Notes on Labor Demand Under A Cobb-Douglas Technology

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1 Cobb-Douglas Production Function

$$Q = Ae^{gt}L^{\alpha}K^{\beta}$$

or

$$ln(Q) = ln(A) + gt + \alpha ln(L) + \beta ln(K)$$

for $g > 0, 0 < \alpha, \beta < 1$, and $0 < \alpha + \beta < 1$. These restrictions describe a CD technology with neutral technological change and decreasing returns to scale. The marginal product expressions are given by

$$\begin{aligned} MP_L &= \frac{\partial Q}{\partial L} \\ &= \alpha Ae^{gt}L^{\alpha-1}K^{\beta} \\ &= \alpha \frac{Q}{L} \\ &= \alpha \left(AP_L \right), \end{aligned}$$

and

$$\begin{aligned} MP_K &= \frac{\partial Q}{\partial K} \\ &= \beta Ae^{gt}L^{\alpha}K^{\beta-1} \\ &= \beta \frac{Q}{K} \\ & \beta \left(AP_K \right). \end{aligned}$$

2 Conditional Input Demand

Conditional input demand functions are obtained from cost minimization. Let w be the marginal cost of an additional unit of labor (e.g. the hourly wage), and let r be the marginal cost/user cost (rental rate on capital) of an additional unit of the non-labor input. The economic problem is formally stated as

$$\underset{(L,K)}{min}\ C = wL + rK \text{ s.t. } Q = Ae^{gt}L^{\alpha}K^{\beta}.$$

The cost minimizing solutions for L and K are more easily obtained from

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

$\Rightarrow$

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{K}{L}\right) = \frac{w}{r}$$

$\Rightarrow$

$$\frac{K}{L} = \left(\frac{\beta}{\alpha}\right)\left(\frac{w}{r}\right).$$

This latter result describes the optimal capital(non-labor input) - labor ratio for a given wage rental rate ratio (expansion path). It follows that

$$K = \left(\frac{\beta}{\alpha}\right)\left(\frac{w}{r}\right)L.$$

This expression can be substituted for K in the production function to solve for L. One can then solve for K. In terms of logs, the conditional input demand functions can be shown to be

$$ln(L) = \left(\frac{-1}{\alpha+\beta}\right)\left\{\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha}\right)\right] + \beta ln\left(\frac{w}{r}\right) + gt - ln(Q)\right\}$$
$$ln(K) = \left(\frac{-1}{\alpha+\beta}\right)\left\{\left[ln(A) - \alpha ln\left(\frac{\beta}{\alpha}\right)\right] - \alpha ln\left(\frac{w}{r}\right) + gt - ln(Q)\right\}.$$

The Cobb-Douglas production function is a special case of the Constant Elasticity of Substitution (CES) production technology. To see this, note that the optimal capital/labor ratio may be expressed in logs as

$$ln\left(\frac{K}{L}\right) = ln\left(\frac{\beta}{\alpha}\right) + ln\left(\frac{w}{r}\right).$$

In general a two-input CES technology implies

$$ln\left(\frac{K}{L}\right) = b + \sigma ln\left(\frac{w}{r}\right).$$

In the special case of a CD technology, $b = ln\left(\frac{\beta}{\alpha}\right)$ and $\sigma = 1$ for a unitary elasticity of substitution.

3 Input Demand Under Long-Run Profit Maximization

Input demand functions under long-run profit maximization with decreasing returns to scale can be derived. Let p be the price of each unit of input sold. In the simplest case one assumes price equals marginal revenue, $p = MR$. The formal maximization problem is usually stated in terms of choosing the optimal output to maximize profits:

$$\max_{Q} \pi = pQ - C(Q).$$

The maximization problem can be stated in terms of optimal inputs:

$$\max_{L,K} \pi = pQ(L,K) - wL - rK.$$

The input demand functions under LR profit maximization can be solved from the marginal revenue product (MRP) conditions:

$$MRP_L = w$$
$$MRP_K = r,$$

where $MRP_L = MR\text{x}MP_L$, and $MRP_K = MR\text{x}MP_K$. Upon substituting p for MR and the expressions derived earlier for MP_L and MP_K, one obtains two equations in two unknowns (L and K). Upon solving these equations and taking logs, one obtains the following input demand functions under LR profit max:

$$ln(L) = \left(\frac{-1}{1-\alpha-\beta}\right)\left\{\left[-ln(\alpha A) - \beta ln\left(\frac{\beta}{\alpha}\right)\right] + (1-\beta)\,ln\left(\frac{w}{p}\right) + \beta ln\left(\frac{r}{p}\right) - gt\right\}$$
$$ln(K) = \left(\frac{-1}{1-\alpha-\beta}\right)\left\{\left[-ln(\beta A) + \alpha ln\left(\frac{\beta}{\alpha}\right)\right] + \alpha ln\left(\frac{w}{p}\right) + (1-\alpha)\,ln\left(\frac{r}{p}\right) - gt\right\}.$$

Note: profit max $\Rightarrow$ cost min because

$$\frac{MR\text{x}MP_L}{MR\text{x}MP_K} = \frac{MP_L}{MP_K} = \frac{w}{r}.$$

However, cost min $\nRightarrow$profit max because $\frac{MP_L}{MP_K} = \frac{w}{r} \nRightarrow MR\text{x}MP_L = w$ and $MR\text{x}MP_K = r$.

4 Input Demand Under Short-Run Profit Maximization

Labor demand under short-run profit max involves a single profit maximizing condition:

$$MRP_L = w$$

$\Rightarrow$

$$MR\text{x}MP_L = w$$

$\Rightarrow$

$$p\text{x}MP_L = w$$

$\Rightarrow$

$$MP_L = \frac{w}{p}.$$

Upon substitution for MP_L and solving for L, we can obtain the demand for labor under SR profit max. Expressed in logs, the labor demand function is given by

$$ln(L) = \left(\frac{1}{1-\alpha}\right)\left[ln(\alpha A) - ln\left(\frac{w}{p}\right) + \beta ln(K) + gt\right].$$

In this case K is being held constant. It is possible to imagine cases in which the labor inputs cannot be changed in the short-run so that only the non-labor inputs are variable. In this scenario the profit maximizing condition and the labor demand function are given by

$$MP_K = \frac{r}{p}$$

and

$$ln(K) = \left(\frac{1}{1-\beta}\right)\left[ln(\beta A) - ln\left(\frac{r}{p}\right) + \alpha ln(L) + gt\right].$$

5 Input Demands For a Public Agency

Input demands for a public agency can be arrived at in an analogous fashion to consumer utility maximization. One can think of a public agency as tasked with maximizing the amount of goods or services provided to the public subject to a budget constraint. In the present context, the decision problem is

$$\underset{L,K}{max}\ Q = Ae^{gt}L^{\alpha}K^{\beta} \text{ s.t. } C = wL + rK,$$

where C is the public agency's budget. This condition implies efficiency, so we may use the cost minimization condition

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

which implies

$$K = \left(\frac{\beta}{\alpha}\right)\left(\frac{w}{r}\right)L.$$

Upon substitution for K in the cost equation $C = wL + rK$, we can solve for L. We can also solve for K in an analogous fashion. The input cost shares are constants:

$$\frac{wL}{C} = \frac{\alpha}{\alpha + \beta}$$
$$\frac{wL}{C} = \frac{\beta}{\alpha + \beta},$$

so that the resulting input demand functions are given by

$$L = \frac{\alpha}{\alpha + \beta}\frac{C}{w}$$
$$K = \frac{\beta}{\alpha + \beta}\frac{C}{r}.$$

In terms of logs, the input demand functions are expressed by

$$
\begin{aligned}
ln\,(L) &= ln\left(\frac{\alpha}{\alpha+\beta}\right) + ln\left(\frac{C}{w}\right) \\
ln\,(K) &= ln\left(\frac{\beta}{\alpha+\beta}\right) + ln\left(\frac{C}{r}\right).
\end{aligned}
$$

Note that there is no effect of non-neutral technological change on the input demands of a public agency nor are there cross input price effects.

6 Output Supply

The cost function corresponding to a Cobb-Douglas technology can be obtained by substituting the conditional input demand functions $L(w, r, Q, t)$ and $K(w, r, Q, t)$ into the cost equation:

$$
\begin{aligned}
C(w, r, Q, t) &= wL(w, r, Q, t) + rK(w, r, Q, t) \\
&= (\psi)\left(w^{\frac{\alpha}{\alpha+\beta}}\right)\left(r^{\frac{\beta}{\alpha+\beta}}\right)\left(Q^{\frac{1}{\alpha+\beta}}\right)\left(e^{\frac{-gt}{\alpha+\beta}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\psi &= \left[A^{\frac{-1}{\alpha+\beta}}\right]\left[\left(\frac{\beta}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] \\
&= \left[\frac{\alpha+\beta}{\beta}\right]\left[A^{\frac{-1}{\alpha+\beta}}\right]\left[\left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right].
\end{aligned}
$$

The marginal cost function for the Cobb-Douglas technology is obtained as

$$
\begin{aligned}
MC(w,r,Q,t) &= \frac{\partial C(w,r,Q,t)}{\partial Q} \\
&= \left(\frac{\psi}{\alpha+\beta}\right)\left(w^{\frac{\alpha}{\alpha+\beta}}\right)\left(r^{\frac{\beta}{\alpha+\beta}}\right)\left(Q^{\frac{1-\alpha-\beta}{\alpha+\beta}}\right)\left(e^{\frac{-gt}{\alpha+\beta}}\right).
\end{aligned}
$$

In a competitive or price taking setting, $MR = p = MC(w,r,Q,t)$. Therefore, the inverse supply function would simply be

$$
p = \left(\frac{\psi}{\alpha+\beta}\right)\left(w^{\frac{\alpha}{\alpha+\beta}}\right)\left(r^{\frac{\beta}{\alpha+\beta}}\right)\left(Q^{\frac{1-\alpha-\beta}{\alpha+\beta}}\right)\left(e^{\frac{-gt}{\alpha+\beta}}\right).
$$

The output supply function is obtained by solving for Q as a function of p:

$$
Q^s = \left(\frac{\alpha+\beta}{\psi}\right)^{\frac{\alpha+\beta}{1-(\alpha+\beta)}}\left(w^{\frac{-\alpha}{1-(\alpha+\beta)}}\right)\left(r^{\frac{-\beta}{1-(\alpha+\beta)}}\right)\left(e^{\frac{gt}{1-(\alpha+\beta)}}\right)\left(p^{\frac{\alpha+\beta}{1-(\alpha+\beta)}}\right).
$$

In logs and after rearranging terms, the output supply function may be expressed as

$$
ln(Q^s) = \left(\frac{1}{1-\alpha-\beta}\right)\left\{[ln(A) + \alpha ln(\alpha) + \beta ln(\beta)] - \alpha ln\left(\frac{w}{p}\right) - \beta ln\left(\frac{r}{p}\right) + gt\right\}.
$$

7 Output Demand Function

To complete the market, we require an output demand function. A simple example that will suffice for illustrative purposes is given by

$$
ln(Q^d) = \theta_0 + \theta_1 ln\left(\frac{p}{y}\right) + \theta_2 t
$$

where y is some measure of consumer income and $\theta_1 < 0$.

8 Market Demand and Supply

We can solve for equilibrium market quantity and price by equating demand and supply:

$$ln(Q^d) = ln(Q^s)$$

$\Rightarrow$

$$\begin{aligned} ln(p) =& \left[\frac{1-\alpha-\beta}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right][-\theta_0+\theta_1 ln(y)-\theta_2 t] \\ &+\left[\frac{1-\alpha-\beta}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left\{\left(\frac{1}{1-\alpha-\beta}\right)[ln(A)+\alpha ln(\alpha)+\beta ln(\beta)]\right. \\ &\left.-\alpha ln(w_t)-\beta ln(r_t)+gt\right\} \\ =& \left[\frac{1-\alpha-\beta}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left\{-\theta_0+\left(\frac{1}{1-\alpha-\beta}\right)[ln(A)+\alpha ln(\alpha)+\beta ln(\beta)]\right. \\ &\left.+\left[\left(\frac{g}{1-\alpha-\beta}\right)-\theta_2\right]t+\theta_1 ln(y)-\left(\frac{\alpha ln(w)+\beta ln(r)}{1-\alpha-\beta}\right)\right\} \end{aligned}$$

$$\begin{aligned} ln(Q) =& \left[\frac{\theta_1(\alpha+\beta)}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left[\frac{ln(A)+\alpha ln(\alpha)+\beta ln(\beta)}{(\alpha+\beta)}-\frac{\theta_0}{\theta_1}\right] \\ &-\left[\frac{\theta_1(\alpha+\beta)}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left[\frac{\alpha ln(w)+\beta ln(r)}{(\alpha+\beta)}\right] \\ &+\left[\frac{\theta_1(\alpha+\beta)}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left(\frac{g}{\alpha+\beta}-\frac{\theta_2}{\theta_1}\right)t \\ &+\left[\frac{\theta_1(\alpha+\beta)}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]ln(y) \\ =& \left[\frac{\theta_1}{\theta_1(1-\alpha-\beta)-(\alpha+\beta)}\right]\left\{ln(A)+\alpha ln(\alpha)+\beta ln(\beta)-\frac{\theta_0}{\theta_1}(\alpha+\beta)\right. \\ &\left.-\alpha ln(w)-\beta ln(r)+\left[g-\frac{\theta_2}{\theta_1}(\alpha+\beta)\right]t+(\alpha+\beta)ln(y)\right\} \end{aligned}$$