Notes on Gauss-Seidel Algorithm

R.L. Oaxaca
University of Arizona

Econometric models of simultaneous equations can be used for forecasts and counterfactual policy simulations. In a linear system the solution to the system is a set of linear reduced form equations. If the system is nonlinear in the parameters, then there is no closed-form solution. In this case one has to rely upon numerical methods. The Gauss-Seidel algorithm is an intuitive method for obtaining numerical solutions for nonlinear, simultaneous equations. Unfortunately, there is no guarantee that a solution exists or that it is unique. The method requires initial guesses at the values of the RHS endogenous variables. In a forecasting exercise, one would typically use the last known values of the endogenous variables as initial values.

1 Example

An example of the Gauss-Seidel algorithm for a hypothetical nonlinear system is given below.

Endogenous Variables: y_{it}, $i = 1, ..., 5$

Exogenous Variables: Z_{it}, $i = 1, 2, 3$

Recursive Endogenous Variables: y_{it}, $i = 1, 2$

Simultaneous Endogenous Variables: y_{it}, $i = 3, 4, 5$

Recursive Block

$y_{1t} = f_1\left(Z_{2t}, Z_{3t}\right)$

$y_{2t} = f_2\left(y_{1t}, Z_{1t}\right)$

Simultaneous Block

$y_{3t} = f_3(y_{1t}, y_{4t}, y_{5t-1}, Z_{2t})$

$y_{4t} = f_4(y_{3t}, y_{4t-1}, Z_{1t}, Z_{3t-1})$

$y_{5t} = f_5(y_{3t}, y_{4t})$

Period T+1 Forecast

$\tilde{y}_{1T+1} = f_1\left(Z_{2T+1}, Z_{3T+1}\right)$

$\tilde{y}_{2T+1} = f_2\left(\tilde{y}_{1T+1}, Z_{1T+1}\right)$

$\tilde{y}^{(1)}_{3T+1} = f_3(\tilde{y}_{1T+1}, \tilde{y}^{(0)}_{4T+1}, y_{5T}, Z_{2T+1})$

$\tilde{y}^{(1)}_{4T+1} = f_4(\tilde{y}^{(1)}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T})$

$\tilde{y}^{(1)}_{5T+1} = f_5(\tilde{y}^{(1)}_{3T+1}, \tilde{y}^{(1)}_{4T+1})$

(typically, $\tilde{y}^{(0)}_{4T+1} = y_{4T}$)

Subsequent iterations apply only to the simultaneous block:

$\tilde{y}^{(2)}_{3T+1} = f_3(\tilde{y}_{1T+1}, \tilde{y}^{(1)}_{4T+1}, y_{5T}, Z_{2T+1})$

$\tilde{y}^{(2)}_{4T+1} = f_4(\tilde{y}^{(2)}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T})$

$\tilde{y}^{(2)}_{5T+1} = f_5(\tilde{y}^{(2)}_{3T+1}, \tilde{y}^{(2)}_{4T+1})$

If convergence is reached on the jth iteration, the forecast for period T + 1 is given by

$\tilde{Y}'_{T+1} = (\tilde{y}_{1T+1}, \tilde{y}_{2T+1}, \tilde{y}_{3T+1}, \tilde{y}_{4T+1}, \tilde{y}_{5T+1})$ where

$\tilde{y}_{1T+1} = f_1\left(Z_{2T+1}, Z_{3T+1}\right)$

$\tilde{y}_{2T+1} = f_2\left(\tilde{y}_{1T+1}, Z_{1T+1}\right)$

$\tilde{y}_{3T+1} = f_3(\tilde{y}_{1T+1}, \tilde{y}^{(j-1)}_{4T+1}, y_{5T}, Z_{2T+1})$

$\tilde{y}_{4T+1} = f_4(\tilde{y}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T})$

$\tilde{y}_{5T+1} = f_5(\tilde{y}_{3T+1}, \tilde{y}_{4T+1})$

$(\tilde{y}_{iT+1} = \tilde{y}^{(j)}_{iT+1} \cong \tilde{y}^{(j-1)}_{iT+1},\ i = 3, 4, 5)$

Period T+2 Forecast

$\tilde{y}_{1T+2} = f_1\left(Z_{2T+2}, Z_{3T+2}\right)$

$\tilde{y}_{2T+2} = f_2\left(\tilde{y}_{1T+2}, Z_{1T+2}\right)$

$\tilde{y}^{(1)}_{3T+2} = f_3(\tilde{y}_{1T+2}, \tilde{y}^{(0)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2})$

$\tilde{y}^{(1)}_{4T+2} = f_4(\tilde{y}^{(1)}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1})$

$\tilde{y}^{(1)}_{5T+2} = f_5(\tilde{y}^{(1)}_{3T+2}, \tilde{y}^{(1)}_{4T+2})$

(typically, $\tilde{y}^{(0)}_{4T+2} = \tilde{y}_{4T+1}$)

Subsequent iterations apply only to the simultaneous block:

$\tilde{y}^{(2)}_{3T+2} = f_3(\tilde{y}_{1T+2}, \tilde{y}^{(1)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2})$

$\tilde{y}^{(2)}_{4T+2} = f_4(\tilde{y}^{(2)}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1})$

$\tilde{y}^{(2)}_{5T+2} = f_5(\tilde{y}^{(2)}_{3T+2}, \tilde{y}^{(2)}_{4T+2})$

etc.

If convergence is reached on the jth iteration, the forecast for period T + 2 is given by

$\tilde{Y}'_{T+2} = (\tilde{y}_{1T+2}, \tilde{y}_{2T+2}, \tilde{y}_{3T+2}, \tilde{y}_{4T+2}, \tilde{y}_{5T+2})$ where

$\tilde{y}_{1T+2} = f_1\left(Z_{2T+2}, Z_{3T+2}\right)$

$\tilde{y}_{2T+2} = f_2\left(\tilde{y}_{1T+2}, Z_{1T+2}\right)$

$\tilde{y}_{3T+2} = f_3(\tilde{y}_{1T+2}, \tilde{y}^{(j-1)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2})$

$\tilde{y}_{4T+2} = f_4(\tilde{y}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1})$

$\tilde{y}_{5T+2} = f_5(\tilde{y}_{3T+2}, \tilde{y}_{4T+2})$

($\tilde{y}_{iT+2} = \tilde{y}^{(j)}_{iT+2} \cong \tilde{y}^{(j-1)}_{iT+2}$, $i = 3, 4, 5$)

In general a forecast for period T + s is given by

$\tilde{Y}'_{T+s} = (\tilde{y}_{1T+s}, \tilde{y}_{2T+s}, \tilde{y}_{3T+s}, \tilde{y}_{4T+s}, \tilde{y}_{5T+s})$ where

$\tilde{y}_{1T+s} = f_1\,(Z_{2T+s}, Z_{3T+s})$

$\tilde{y}_{2T+s} = f_2\,(\tilde{y}_{1T+s}, Z_{1T+s})$

$\tilde{y}_{3T+s} = f_3(\tilde{y}_{1T+s}, \tilde{y}^{(j-1)}_{4T+s}, \tilde{y}_{5T+s-1}, Z_{2T+s})$

$\tilde{y}_{4T+s} = f_4(\tilde{y}_{3T+s}, \tilde{y}_{4T+s-1}, Z_{1T+s}, Z_{3T+s-1})$

$\tilde{y}_{5T+s} = f_5(\tilde{y}_{3T+s}, \tilde{y}_{4T+s})$

$(\tilde{y}_{iT+s} = \tilde{y}^{(j)}_{iT+s} \cong \tilde{y}^{(j-1)}_{iT+s},\ i = 3, 4, 5)$

Gauss-Seidel Example

(1) $Y_1 = a_0 + a_1 Y_2$

(2) $Y_2 = b_0 + b_1 Y_1$

Solution:

(3) $$Y_1^* = \frac{a_0 + a_1 b_0}{1 - a_1 b_1},$$

(4) $$Y_2^* = \frac{b_0 + b_1 a_0}{1 - a_1 b_1} \qquad \text{for } a_1 b_1 \neq 1.$$

Gauss-Seidel solution method

For the jth iteration the model is specified as

(5) $Y_1^{(j)} = a_0 + a_1 Y_2^{(j-1)}$

(6) $Y_2^{(j)} = b_0 + b_1 Y_1^{(j)}$.

Upon substitution of (5) into (6) we have

(7) $$\boxed{Y_2^{(j)} = b_0 + b_1 a_0 + a_1 b_1 Y_2^{(j-1)}}$$

which is a first-order linear difference equation.

Accordingly, the solution to (7) is

(8) $$\boxed{Y_2^{(j)} = \left[Y_2^{(0)} - \frac{b_0 + b_1 a_0}{1 - a_1 b_1}\right](a_1 b_1)^j + \frac{b_0 + b_1 a_0}{1 - a_1 b_1}}$$

$$\lim_{j\to\infty} Y_2^{(j)} = \frac{b_0 + b_1 a_0}{1 - a_1 b_1} \quad (\text{for } |a_1 b_1| < 1)$$

$$= Y_2^*.$$

Alternatively, the solution algorithm can be expressed as

(9) $Y_2^{(j)} = b_0 + b_1 Y_1^{(j-1)}$

(10) $Y_1^{(j)} = a_0 + a_1 Y_2^{(j)}$

Upon substitution of (9) into (10) we have

(11) $\boxed{Y_1^{(j)} = a_0 + a_1 b_0 + a_1 b_1 Y_1^{(j-1)}}$

which is a first-order linear difference equation.

Accordingly, the general solution to (11) is

(12) $\boxed{Y_1^{(j)} = \left\{Y_1^{(0)} - \frac{a_0 + a_1 b_0}{1 - a_1 b_1}\right\}(a_1 b_1)^j + \frac{a_0 + a_1 b_0}{1 - a_1 b_1}}$

where

$$\lim_{j\to\infty} Y_1^{(j)} = \frac{a_0 + a_1 b_0}{1 - a_1 b_1} \quad (\text{for } |a_1 b_1| < 1)$$

$$= Y_1^*.$$

Suppose $|a_1 b_1| > 1$. In this case (8) and (12) will not converge. A simple re-normalization of (1) and (2) will solve the problem:

$$Y_2 = b_0' + b_1' Y_1$$

$$Y_1 = a_0' + a_1' Y_2$$

where $b_0' = -a_0/a_1$, $b_1' = 1/a_1$, $a_0' = -b_0/b_1$, and $a_1' = 1/b_1$.

It is easily seen that $|a_1'b_1'| = |1/a_1 b_1| < 1$.

Numerical example

Let $a_0 = 25$, $a_1 = 1.5$, $b_0 = -22$, $b_1 = 0.8$ (note that $|a_1 b_1| = 1.2 > 1$).

The solution values are $Y_1^* = 40$, $Y_2^* = 10$.

The Gauss-Seidel solution method can be implemented by substituting the appropriate parameter values into (5) and (6) to obtain

$$Y_1^{(j)} = 25 + 1.5Y_2^{(j-1)} \text{ and}$$

$$Y_2^{(j)} = -22 + 0.8Y_1^{(j)} ;$$

or by making the appropriate parameter substitutions in (7) to obtain

(13) $Y_2^{(j)} = -2 + 1.2Y_2^{(j-1)}$.

Let $Y_2^{(0)} = 0$ as an initial condition, then appropriate substitution into (8) yields

(14) $\boxed{Y_2^{(j)} = (-10)(1.2)^j + 10}$

The first-order linear difference equation corresponding to $Y_1^{(j)}$ is obtained by making the appropriate parameter substitutions in (11):

(15) $Y_1^{(j)} = -8 + 1.2Y_1^{(j-1)}$.

Given the initial condition $Y_2^{(0)} = 0$, implies the initial condition $Y_1^{(0)} = 27.5$. Accordingly, appropriate substitution into (12) yields the general solution to (15):

(16) $$\boxed{Y_1^{(j)} = (-12.5)(1.2)^j + 40}$$

j	$Y_1^{(j)}$	$Y_2^{(j)}$
0	27.5	0
1	25	-2
2	22	-4.4
3	18.4	-7.28
4	14.08	-10.736
.	.	.
.	.	.
.	.	.
∞	$-\infty$	$-\infty$

2 Policy Simulation

The Gauss-Seidel example serves as a nice illustration of how one might conduct policy simulations with nonlinear models. We will let the variable Z_{2t} serve as the policy variable which is exogenously changed in period t. How do we estimate the impact of the policy change in Z_{2t} on the endogenous variables in the system? In period t the estimated model together with the equation residuals may expressed as

$$
\begin{aligned}
y_{1t} &= f_1\left(Z_{2t}, Z_{3t}, \hat{\beta}\right) + \hat{u}_{1t} \\
&= \hat{y}_{1t} + \hat{u}_{1t} \\
y_{2t} &= f_2\left(y_{1t}, Z_{1t}, \hat{\beta}\right) + \hat{u}_{2t} \\
&= \hat{y}_{2t} + \hat{u}_{2t} \\
y_{3t} &= f_3(y_{1t}, y_{4t}, y_{5t-1}, Z_{2t}, \hat{\beta}) + \hat{u}_{3t} \\
&= \hat{y}_{3t} + \hat{u}_{3t} \\
y_{4t} &= f_4(y_{3t}, y_{4t-1}, Z_{1t}, Z_{3t-1}, \hat{\beta}) + \hat{u}_{4t} \\
&= \hat{y}_{4t} + \hat{u}_{4t} \\
y_{5t} &= f_5(y_{3t}, y_{4t}, \hat{\beta}) + \hat{u}_{5t} \\
&= \hat{y}_{5t} + \hat{u}_{5t},
\end{aligned}
$$

where $\hat{u}_{jt} = y_{jt} - \hat{y}_{jt}$, $\hat{y}_{jt} = f_j(\cdot)$, and $\hat{\beta}$ is a vector of all of the estimated parameters in the model. Note that not all elements of β appear in every equation.

Now suppose we wish to estimate the impact of a policy change in variable Z_{2t}. We can represent the policy change in variable Z_{2t} as $\Delta Z_{2t} = Z_{2t}^p - Z_{2t}$, where Z_{2t}^p is the counterfactual value of Z_{2t} under the assumed policy. One could simply include the equation residuals and use the Gauss-Seidel algorithm to solve the model with Z_{2t}^p substituted for the actual value Z_{2t}:

$$
\begin{aligned}
y^p_{1t} &= f_1\left(Z^p_{2t}, Z_{3t}, \hat{\beta}\right) + \hat{u}_{1t} \\
y^p_{2t} &= f_2\left(y^p_{1t}, Z_{1t}, \hat{\beta}\right) + \hat{u}_{2t} \\
y^p_{3t} &= f_3(y^p_{1t}, y^p_{4t}, y_{5t-1}, Z^p_{2t}, \hat{\beta}) + \hat{u}_{3t} \\
y^p_{4t} &= f_4(y^p_{3t}, y_{4t-1}, Z_{1t}, Z_{3t-1}, \hat{\beta}) + \hat{u}_{4t} \\
y^p_{5t} &= f_5(y^p_{3t}, y^p_{4t}, \hat{\beta}) + \hat{u}_{5t}.
\end{aligned}
$$

Adding the residuals for the policy simulation corrects for equation error. Since we are interested in a policy simulation in period t, we have not dynamically solved the model for the lagged values of the endogenous variables and simply use their historical lagged values in period $t-1$, i.e. y_{4t-1} and y_{5t-1}. The addition of the in-sample equation residuals to the policy simulation solution allows one to estimate the policy effects of ΔZ_{2t} using actual history as the baseline or control simulation:

$$
\Delta y^p_{jt} = y^p_{jt} - y_{jt}, \;\; j = 1, ..., 5. \tag{1}
$$

An alternative simulation methodology is to set the residuals to 0 and compare the model solutions with and without the policy change. The control solution is obtained from:

$$
\begin{aligned}
\tilde{y}_{1t} &= f_1\left(Z_{2t}, Z_{3t}, \hat{\beta}\right) = \hat{y}_{1t} \\
\tilde{y}_{2t} &= f_2\left(\tilde{y}_{1t}, Z_{1t}, \hat{\beta}\right) = \hat{y}_{2t} \\
\tilde{y}_{3t} &= f_3(\tilde{y}_{1t}, \tilde{y}_{4t}, y_{5t-1}, Z_{2t}, \hat{\beta}) \\
\tilde{y}_{4t} &= f_4(\tilde{y}_{3t}, y_{4t-1}, Z_{1t}, Z_{3t-1}, \hat{\beta}) \\
\tilde{y}_{5t} &= f_5(\tilde{y}_{3t}, \tilde{y}_{4t}, \hat{\beta}).
\end{aligned}
$$

Next, the policy simulation is obtained from the solution of the model using Z_{2t}^p in lieu of Z_{2t}:

$$\begin{aligned}
\tilde{y}_{1t}^p &= f_1\left(Z_{2t}^p, Z_{3t}, \hat{\beta}\right) \\
\tilde{y}_{2t}^p &= f_2\left(\tilde{y}_{1t}^p, Z_{1t}, \hat{\beta}\right) \\
\tilde{y}_{3t}^p &= f_3(\tilde{y}_{1t}^p, \tilde{y}_{4t}^p, y_{5t-1}, Z_{2t}^p, \hat{\beta}) \\
\tilde{y}_{4t}^p &= f_4(\tilde{y}_{3t}^p, y_{4t-1}, Z_{1t}, Z_{3t-1}, \hat{\beta}) \\
\tilde{y}_{5t}^p &= f_5(\tilde{y}_{3t}^p, \tilde{y}_{4t}^p, \hat{\beta}).
\end{aligned}$$

Here the policy effects of ΔZ_{2t} are obtained by using predicted history as the baseline or control simulation:

$$\widetilde{\Delta y}_{jt}^p = \tilde{y}_{jt}^p - \tilde{y}_{jt},\ \ j = 1, ..., 5. \tag{2}$$

With the exception of linear models, in general $\widetilde{\Delta y}_{jt}^p \neq \Delta y_{jt}^p$. Both simulation methodologies are controlled computer experiments in which the only difference between the control and the policy simulations is the policy change in Z_{2t}. Although both simulation approaches are valid, using actual history as the control simulation seems more natural or intuitive.