Notes on Labor Demand Under a CES Technology

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1 CES Production Function

The Constant Elasticity of Substitution (CES) production function for two inputs may be expressed as

$$
\begin{aligned}
Q &= A\left[\alpha L^{\rho} + (1-\alpha)K^{\rho}\right]^{\frac{\phi}{\rho}} e^{gt} \\
&= A\left[\alpha L^{\frac{\sigma-1}{\sigma}} + (1-\alpha)K^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\phi\sigma}{\sigma-1}} e^{gt}
\end{aligned}
$$

for $g, \phi, \sigma > 0, 0 < \alpha < 1$, and $\rho < 1$. The parameter ϕ is the returns to scale parameter:

$$
\begin{aligned}
&\phi = 1 \Rightarrow \text{ constant returns to scale} \\
&0 < \phi < 1 \Rightarrow \text{ decreasing returns to scale} \\
&\phi > 1 \Rightarrow \text{ increasing returns to scale.}
\end{aligned}
$$

The parameter σ is the elasticity of substitution and α is the share parameter. The relationship between σ and ρ can be described by

$$
\begin{aligned}
\rho &= \frac{\sigma-1}{\sigma} \text{ and} \\
\sigma &= \frac{1}{1-\rho}.
\end{aligned}
$$

The CD production function is a limiting case of the CES production function as $\sigma \rightarrow 1$, or equivalently as $\rho \rightarrow 0$.

The marginal product expressions are given by

$$\begin{aligned} MP_L &= \frac{\partial Q}{\partial L} \\ &= \alpha\phi A\left[\alpha L^{\rho} + (1-\alpha)K^{\rho}\right]^{\frac{\phi}{\rho}-1} L^{\rho-1}e^{gt}. \end{aligned}$$

Note that

$$\left[\alpha L^{\rho} + (1-\alpha)K^{\rho}\right] = \left[\frac{Q}{A}e^{-gt}\right]^{\frac{\rho}{\phi}}$$

$\Rightarrow$

$$\left[\alpha L^{\rho} + (1-\alpha)K^{\rho}\right]^{\frac{\phi}{\rho}-1} = \left[\frac{Q}{A}e^{-gt}\right]^{1-\frac{\rho}{\phi}}.$$

Upon appropriate substitution in the marginal product expression, we arrive at

$$MP_L = \alpha\phi L^{\rho-1}A^{\frac{\rho}{\phi}}e^{\frac{g\rho t}{\phi}}Q^{1-\frac{\rho}{\phi}}.$$

Similarly, the marginal product expression for the nonlabor inputs is specified as

$$\begin{aligned} MP_K &= \frac{\partial Q}{\partial K} \\ &= (1-\alpha)\,\phi K^{\rho-1}A^{\frac{\rho}{\phi}}e^{\frac{g\rho t}{\phi}}Q^{1-\frac{\rho}{\phi}} \end{aligned}$$

2 Cost Minimization

The cost minimizing condition leads directly to the relative input demand function:

$$\frac{MP_L}{MP_K} = \frac{w}{r}$$

$\Rightarrow$

$$\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{K}{L}\right)^{1-\rho} = \frac{w}{r}$$

$\Rightarrow$

$$\begin{aligned}\frac{K}{L} &= \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{1-\rho}}\left(\frac{w}{r}\right)^{\frac{1}{1-\rho}} \\ &= \left(\frac{1-\alpha}{\alpha}\right)^{\sigma}\left(\frac{w}{r}\right)^{\sigma}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}ln\left(\frac{K}{L}\right) &= \frac{1}{1-\rho}ln\left(\frac{1-\alpha}{\alpha}\right) + \frac{1}{1-\rho}ln\left(\frac{w}{r}\right) \\ &= \sigma ln\left(\frac{1-\alpha}{\alpha}\right) + \sigma ln\left(\frac{w}{r}\right).\end{aligned}$$

The CD is the special case in which $\sigma = 1$.

Empirically, one could estimate the relative demand equation by OLS if $ln\left(\frac{w}{r}\right)$ were exogenous:

$$ln\left(\frac{K_t}{L_t}\right) = b + \sigma ln\left(\frac{w_t}{r_t}\right) + \varepsilon_{klt}$$

The parameters σ and α are identified, where $b = \sigma ln\left(\frac{1-\alpha}{\alpha}\right)$. One can show that $\alpha = \frac{1}{exp\left[\left(\frac{b}{\sigma}\right)+1\right]}$. Therefore, α can be consistently estimated from

$$\tilde{\alpha} = \frac{1}{exp\left[\left(\frac{\hat{b}^{ols}}{\hat{\sigma}^{ols}}\right)+1\right]}.$$

Of course the parameters ϕ and g are not identified from the relative input demand model.

3 Input Demand Functions, Cost Functions, and Output Supply

The derivations of the input demand functions, cost functions, and output supply proceed in exactly the same way as with the CD technology except that the resulting functions will be more complicated.

As an example, consider the conditional input demand functions for cost minimization. From the optimal capital-labor ratio (relative input demand) expression one can obtain the optimal level of K:

$$\begin{aligned} K &= \left[\left(\frac{1-\alpha}{\alpha} \right)^{\frac{1}{1-\rho}} \left(\frac{w}{r} \right)^{\frac{1}{1-\rho}} \right] L \\ &= \left[\left(\frac{1-\alpha}{\alpha} \right)^{\sigma} \left(\frac{w}{r} \right)^{\sigma} \right] L. \end{aligned}$$

This expression can then be substituted for K in the production function. Upon collecting terms, taking logs, and solving for $ln(L)$, one obtains the conditional input demand function for labor:

$$\begin{aligned} ln(L) &= \left(\frac{1}{\phi} \right) [-ln(A) + ln(Q) - gt] - \frac{1}{\rho} ln \left[\alpha + (1-\alpha) \left(\frac{1-\alpha}{\alpha} \right)^{\frac{\rho}{1-\rho}} \left(\frac{w}{r} \right)^{\frac{\rho}{1-\rho}} \right] \\ &= \left(\frac{1}{\phi} \right) [-ln(A) + ln(Q) - gt] - \frac{\sigma}{\sigma - 1} ln \left[\alpha + (1-\alpha) \left(\frac{1-\alpha}{\alpha} \right)^{\sigma-1} \left(\frac{w}{r} \right)^{\sigma-1} \right]. \end{aligned}$$

Similarly, the conditional input demand function for the nonlabor inputs can be shown to be

$$ln(K) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q) - gt] - \frac{1}{\rho} ln \left[(1-\alpha) + \alpha \left(\frac{1-\alpha}{\alpha}\right)^{\frac{-\rho}{1-\rho}} \left(\frac{w}{r}\right)^{\frac{-\rho}{1-\rho}}\right]$$
$$= \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q) - gt] - \frac{\sigma}{\sigma - 1} ln \left[(1-\alpha) + \alpha \left(\frac{1-\alpha}{\alpha}\right)^{1-\sigma} \left(\frac{w}{r}\right)^{1-\sigma}\right].$$

If Q and $\frac{w}{r}$ were exogenous, one could estimate the conditional input demand functions by NLSUR in terms of the original production function parameters:

$$ln(L_t) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q_t) - gt] - \frac{\sigma}{\sigma - 1} ln \left[\alpha + (1-\alpha) \left(\frac{1-\alpha}{\alpha}\right)^{\sigma-1} \left(\frac{w_t}{r_t}\right)^{\sigma-1}\right] + u_{lt}$$

$$ln(K_t) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q_t) - gt] - \frac{\sigma}{\sigma - 1} ln \left[(1-\alpha) + \alpha \left(\frac{1-\alpha}{\alpha}\right)^{1-\sigma} \left(\frac{w_t}{r_t}\right)^{1-\sigma}\right] + u_{kt},$$

where

$$\begin{pmatrix} u_{lt} \\ u_{kt} \end{pmatrix} \sim N \begin{pmatrix} \sigma_{ll} & \sigma_{lk} \\ \sigma_{kl} & \sigma_{kk} \end{pmatrix}.$$

An alternative estimation strategy is to first estimate the relative input demand function by OLS to obtain consistent estimators of α and σ. Then one can estimate ϕ and g by applying NLSUR to the following input demand equations:

$$ln(Z_{lt}) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q_t) - gt] + u^*_{lt}$$

$$ln(Z_{kt}) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q_t) - gt] + u^*_{kt},$$

where

$$ln(Z_{lt}) = ln(L_t) + \frac{\hat{\sigma}^{ols}}{\hat{\sigma}^{ols} - 1} ln \left[\tilde{\alpha} + (1-\tilde{\alpha}) \left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}}\right)^{\hat{\sigma}^{ols}-1} \left(\frac{w_t}{r_t}\right)^{\hat{\sigma}^{ols}-1}\right]$$

$$ln(Z_{kt}) = ln(K_t) + \frac{\hat{\sigma}^{ols}}{\hat{\sigma}^{ols} - 1} ln \left[(1-\tilde{\alpha}) + \tilde{\alpha} \left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}}\right)^{1-\hat{\sigma}^{ols}} \left(\frac{w_t}{r_t}\right)^{1-\hat{\sigma}^{ols}}\right]$$

$$u_{lt}^{*} = \frac{\hat{\sigma}^{ols}}{\hat{\sigma}^{ols}-1} ln \left[\tilde{\alpha} + (1-\tilde{\alpha}) \left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}} \right)^{\hat{\sigma}^{ols}-1} \left(\frac{w_t}{r_t} \right)^{\hat{\sigma}^{ols}-1} \right]$$
$$- \frac{\sigma}{\sigma-1} ln \left[\alpha + (1-\alpha) \left(\frac{1-\alpha}{\alpha} \right)^{\sigma-1} \left(\frac{w_t}{r_t} \right)^{\sigma-1} \right] + u_{lt}$$

$$u_{kt}^{*} = \frac{\hat{\sigma}^{ols}}{\hat{\sigma}^{ols}-1} ln \left[(1-\tilde{\alpha}) + \tilde{\alpha} \left(\frac{1-\tilde{\alpha}}{\tilde{\alpha}} \right)^{1-\hat{\sigma}^{ols}} \left(\frac{w_t}{r_t} \right)^{1-\hat{\sigma}^{ols}} \right]$$
$$- \frac{\sigma}{\sigma-1} ln \left[(1-\alpha) + \alpha \left(\frac{1-\alpha}{\alpha} \right)^{1-\sigma} \left(\frac{w_t}{r_t} \right)^{1-\sigma} \right] + u_{kt}.$$

Apart from calculation of the asymptotic standard errors, an equivalent formulation of this estimation strategy is given by

$$ln(Z_{lt}) = \gamma_0 + \gamma_1 ln(Q_t) + \gamma_2 t + u_{lt}^{*}$$

$$ln(Z_{kt}) = \gamma_0 + \gamma_1 ln(Q_t) + \gamma_2 t + u_{kt}^{*}$$

where

$\gamma_0 = -\frac{ln(A)}{\phi}$

$\gamma_1 = \frac{1}{\phi}$

$\gamma_2 = -\frac{g}{\phi}.$

It should be clear that the CES parameters A, ϕ, and g are identified and can be consistently estimated from

$$\tilde{A} = exp\left(\frac{-\hat{\gamma}_0^{nlsur}}{\hat{\gamma}_1^{nlsur}} \right)$$

$$\tilde{\phi} = \frac{1}{\hat{\gamma}_1^{nlsur}}$$

$$\tilde{g} = -\frac{\tilde{\gamma}_2^{nlsur}}{\hat{\gamma}_1^{nlsur}}.$$

Suppose data on K_t are not available. One could use NLS to estimate the conditional input demand function for labor and recover all of the parameters of the CES production function:

$$ln(L_t) = \left(\frac{1}{\phi}\right)[-ln(A) + ln(Q_t) - gt] - \frac{\sigma}{\sigma - 1} ln\left[\alpha + (1-\alpha)\left(\frac{1-\alpha}{\alpha}\right)^{\sigma-1}\left(\frac{w_t}{r_t}\right)^{\sigma-1}\right] + u_{lt}.$$