Notes on Partial Elasticities of Substitution

Econ 696i

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Hicks-Allen Partial Elasticities of Substitution

When there are more than 2 inputs, the imposition of a constant elasticity of substitution across all pairs of inputs is restrictive. The Hicks-Allen partial elasticities of substitution is one means of introducing flexibility that allows for elasticities of substitution to vary across pairs of inputs and to vary across time. To begin, we will denote the cost function by $C(w_1, w_{2,}..., w_j, Q)$, where w_i is the ith input price, and Q is output. The usual restrictions on the cost function are assumed. By Shepard's lemma we have the result that

$$\begin{aligned} X_i &= \frac{\partial C}{\partial w_i} \\ &= C_i, \end{aligned}$$

where $X_i = f_i(w_1, w_{2,}..., Q)$ is the conditional input demand function for input i. The cross-elasticity of demand is given by

$$\begin{aligned} \eta_{ij} &= \frac{w_j}{X_i}\frac{\partial X_i}{\partial w_j} \\ &= \frac{w_j}{C_i}C_{ij}, \end{aligned}$$

where $C_{ij} = \frac{\partial X_i}{\partial w_j} = \frac{\partial C_i}{\partial w_j} = \frac{\partial C_j}{\partial w_i} = \frac{\partial X_j}{\partial w_i} = C_{ji}$. The partial elasticity of substitution σ_{ij} is obtained by weighting the cross-elasticity of demand by the inverse of the jth input's share of total cost:

$$\begin{aligned}\sigma_{ij} &= \frac{\eta_{ij}}{s_j} \\ &= \frac{CC_{ij}}{C_i C_j} = \frac{CC_{ji}}{C_i C_j} = \sigma_{ji},\end{aligned}$$

where $s_j = \frac{w_j X_j}{C}$. In the presence of more than 2 inputs, $\sigma_{ij} \gtrless 0$.

It turns out that

$$\begin{aligned}C_{ij} &= \frac{\partial X_i}{\partial w_j} \\ &= \left(\frac{X_i X_j}{C}\right)\left[\frac{\frac{\partial s_i}{\partial ln(w_j)} + s_i s_j}{s_i s_j}\right]. \end{aligned} \tag{1}$$

To see this note that

$$
\begin{aligned}
\frac{\partial s_i}{\partial ln(w_j)} &= \frac{\partial s_i}{\partial w_j} \cdot \frac{\partial w_j}{\partial ln(w_j)} \\
&= \frac{\frac{\partial s_i}{\partial w_j}}{\frac{\partial ln(w_j)}{\partial w_j}} \\
&= \frac{\frac{\partial s_i}{\partial w_i}}{\frac{1}{w_i}} \\
&= w_j \frac{\partial s_i}{\partial w_j} \\
&= w_j \frac{\partial \left(\frac{w_i X_i}{C} \right)}{\partial w_j} \\
&= \frac{w_j}{C^2} \left[C \frac{\partial \left(w_i X_i \right)}{\partial w_j} - w_i X_i \frac{\partial C}{\partial w_j} \right] \\
&= \frac{w_j}{C^2} \left[C \left(w_i \frac{\partial X_i}{\partial w_j} \right) - w_i X_i X_j \right] \\
&= \frac{w_j w_i}{C} \frac{\partial X_i}{\partial w_j} - \frac{w_j w_i X_i X_j}{C^2} \\
&= \frac{w_j w_i}{C} \frac{\partial X_i}{\partial w_j} - s_i s_j \qquad (2)
\end{aligned}
$$

Next substitute (2) for $\frac{\partial s_i}{\partial ln(w_j)}$ in (1) and collect terms to show

$$
\left(\frac{X_i X_j}{C} \right) \left[\frac{\frac{\partial s_i}{\partial ln(w_j)} + s_i s_j}{s_i s_j} \right] = \frac{\partial X_i}{\partial w_j}.
$$

Note that

$$\begin{aligned}\frac{C}{C_iC_j} &= \frac{C}{X_iX_j} \\ &= \frac{w_jw_iC^2}{Cw_jw_iX_iX_j} \\ &= \frac{w_iw_j}{Cs_is_j}.\end{aligned}$$

From simple substitutions for $\dfrac{C}{C_iC_j}$ and C_{ij}, we obtain

$$\begin{aligned}\sigma_{ij} &= \frac{CC_{ij}}{C_iC_j} \\ &= \left(\frac{w_iw_j}{Cs_is_j}\right)\left(\frac{X_iX_j}{C}\right)\left[\frac{\dfrac{\partial s_i}{\partial ln(w_j)} + s_is_j}{s_is_j}\right] \\ &= \frac{\dfrac{\partial s_i}{\partial ln(w_j)} + s_is_j}{s_is_j}. \qquad (3)\end{aligned}$$

The own partial elasticity of substitution can be derived in a similar fashion. The own (conditional) input elasticity of demand is given by

$$\begin{aligned}\eta_{ii} &= \frac{w_i}{X_i}\frac{\partial X_i}{\partial w_i} < 0 \\ &= \frac{w_i}{C_i}C_{ii},\end{aligned}$$

where $C_{ii} = \dfrac{\partial C_i}{\partial w_i} = \dfrac{\partial X_i}{\partial w_i}$. The partial own elasticity of substitution σ_{ii} is obtained by weighting the own elasticity of demand by the inverse of the own input's share of total cost:

$$\begin{aligned}\sigma_{ii} &= \frac{\eta_{ii}}{s_i} \\ &= \frac{CC_{ii}}{C_i^2} < 0.\end{aligned}$$

It turns out that

$$\begin{aligned} C_{ii} &= \frac{\partial X_i}{\partial w_i} \\ &= \left(\frac{C}{w_i^2}\right)\left[\frac{\partial s_i}{\partial ln(w_i)} + (s_i)(s_i - 1)\right]. \end{aligned} \tag{4}$$

To see this note that

$$\begin{aligned} \frac{\partial s_i}{\partial ln(w_i)} &= \frac{\partial s_i}{\partial w_i} \cdot \frac{\partial w_i}{\partial ln(w_i)} \\ &= \frac{\frac{\partial s_i}{\partial w_i}}{\frac{\partial ln(w_i)}{\partial w_i}} \\ &= \frac{\frac{\partial s_i}{\partial w_i}}{\frac{1}{w_i}} \\ &= w_i \frac{\partial s_i}{\partial w_i} \\ &= w_i \frac{\partial \left(\frac{w_i X_i}{C}\right)}{\partial w_i} \\ &= \frac{w_i}{C^2}\left[C \frac{\partial (w_i X_i)}{\partial w_i} - w_i X_i \frac{\partial C}{\partial w_i}\right] \\ &= \frac{w_i}{C^2}\left[C\left(w_i \frac{\partial X_i}{\partial w_i} + X_i\right) - w_i X_i^2\right] \\ &= \frac{w_i^2}{C}\frac{\partial X_i}{\partial w_i} + \frac{w_i X_i}{C} - \frac{w_i^2 X_i^2}{C} \\ &= \frac{w_i^2}{C}\frac{\partial X_i}{\partial w_i} + s_i - s_i^2 \\ &= \frac{w_i^2}{C}\frac{\partial X_i}{\partial w_i} - (s_i)(s_i - 1) \end{aligned} \tag{5}$$

Next substitute (5) for $\frac{\partial s_i}{\partial ln(w_i)}$ in (4) and collect terms to show

$$\left(\frac{C}{w_i^2}\right)\left[\frac{\partial s_i}{\partial ln(w_i)} + (s_i)(s_i - 1)\right] = \frac{\partial X_i}{\partial w_i}.$$

Note that

$$\begin{aligned}\frac{C}{C_i^2} &= \frac{C}{X_i^2} \\ &= \frac{w_i^2 C^2}{C w_i^2 X_i^2} \\ &= \frac{w_i^2}{C s_i^2}.\end{aligned}$$

From simple substitutions for $\frac{C}{C_i^2}$ and C_{ii}, we obtain

$$\begin{aligned}\sigma_{ii} &= \frac{C C_{ii}}{C_i^2} \\ &= \left(\frac{w_i^2}{C s_i^2}\right)\left(\frac{C}{w_i^2}\right)\left[\frac{\partial s_i}{\partial ln(w_i)} + (s_i)(s_i - 1)\right] \\ &= \frac{\frac{\partial s_i}{\partial ln(w_i)} + (s_i)(s_i - 1)}{s_i^2}.\end{aligned} \tag{6}$$

Translog Cost Function

A second order approximation to the cost function is given by the Translog Cost Function:

$$\ln(C) = \beta_0 + \sum_{i=1}^{J} \beta_i \ln(w_i) \;+ \frac{1}{2}\sum_{i=1}^{J}\sum_{j=1}^{J} \delta_{ij} \ln(w_i) \ln(w_j) \;+ \theta \ln(Q)$$

where θ is the returns to scale parameter such that $\theta = 1$ implies constant returns to scale, $\theta < 1$ implies increasing returns to scale, and $\theta > 1$ implies decreasing returns to scale. Homogeneity and aggregation conditions imply the following parameter restrictions:

$$\sum_{i=1}^{J} \beta_i = 1$$

$$\sum_{i=1}^{J} \delta_{ij} = 0$$

$$\sum_{i=j}^{J} \delta_{ij} = 0$$

$$\delta_{ij} = \delta_{ji}$$

As an example, suppose $J = 3$ inputs. Without loss of generality let input 3 refer to the non-labor inputs where $w_3 = r$. By differentiating the cost function with respect to the $\ln(w_i)'s$, applying Shepard's lemma, and imposing the parameter restrictions, we obtain the cost share equations:

$$s_1 = \beta_1 + \delta_{11} \ln\left(\frac{w_1}{r}\right) + \delta_{12} \ln\left(\frac{w_2}{r}\right)$$

$$s_2 = \beta_2 + \delta_{21} \ln\left(\frac{w_1}{r}\right) + \delta_{22} \ln\left(\frac{w_2}{r}\right)$$

$$s_3 = \beta_3 + \delta_{31} \ln\left(\frac{w_1}{r}\right) + \delta_{32} \ln\left(\frac{w_2}{r}\right) \;,$$

where $s_1 = \left(\dfrac{w_1 L_1}{w_1 L_1 + w_2 L_2 + rK}\right)$, $s_2 = \left(\dfrac{w_2 L_2}{w_1 L_1 + w_2 L_2 + rK}\right)$,
and $s_3 = \left(\dfrac{rK}{w_1 L_1 + w_2 L_2 + rK}\right)$.

Since the equations add to 1, one could drop the third equation and estimate using 3SLS with cross-equation restrictions on the parameters. The variance-covariance matrix of the disturbances is not singular since we are dropping one of the equations.

The Hicks-Allen partial elasticities of substitution for the Translog Cost function are given below. In the case of the cross-elasticity, we have

$$\begin{aligned}\sigma_{ij} &= \frac{CC_{ij}}{C_i C_j} \\ &= \frac{\frac{\partial s_i}{\partial ln(w_j)} + s_i s_j}{s_i s_j} \\ &= \frac{\delta_{ij} + s_i s_j}{s_i s_j},\ i \neq j.\end{aligned}$$

The own elasticity of substitution is given by

$$\begin{aligned}\sigma_{ii} &= \frac{CC_{ii}}{C_i^2} \\ &= \frac{\frac{\partial s_i}{\partial ln(w_i)} + (s_i)(s_i - 1)}{s_i^2} \\ &= \frac{\delta_{ii} + s_i(s_i - 1)}{s_i^2} < 0.\end{aligned}$$

The original conditional input demand elasticities can easily be recovered:

$$\eta_{ij} = s_j \sigma_{ij},\ i \neq j$$

$$\eta_{ii} = s_i \sigma_{ii},\ i = j.$$