The Economics of Employment and Hours Decisions of Firms

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Abstract

These notes introduce the basic concepts about employer choices over employment and the work week. The concepts are illustrated with a simple Cobb-Douglas technology. The C-D example is used to demonstrate both the theoretical application and the empiricical estimation issues associated with modeling labor demand.

I. INTRODUCTION

Background reading:

Cahuc, Pierre and André Zylberberg (2004), *Labor Economics*, MIT Press, Chapter 4, pp. 193-205

Hart, Robert A. (2004), *The Economics of Working Overtime*, Cambridge University Press.

Bauer, Thomas and Klaus F. Zimmermann (1999), "Overtime Work and Overtime Compensation in Germany", *Scottish Journal of Political Economy,* 46(4), pp. 419-36.

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When confronted with setting the hours for the workweek and the number of workers to employ, firms clearly are not indifferent in this choice. For example, imagine that a firm requires 400 person-hours per week. This employment demand could be satisfied by employing 10 workers for 40 hours each. But the same demand could be satisfied by employing 5 workers for 80 hours each. Or even employing 400 workers for 1 hour each, and several other combinations as well.

What is the source of an employer's lack of indifference between the number of workers to be employed and the number of hours each individual works? There are two reasons why employers would care about the mix of employment and hours. First of all, there are physical, legal, and logical limitations on the number of hours an individual can work in a week. Secondly, there are distinctly different costs associated with an additional hour added to the work week versus adding another worker to the payroll. Some of the cost differences may be induced by legal requirements regarding compensation for overtime work.

The table below illustrates overtime hours laws and agreements for different European countries.

Table 1: **Principal features of overtime schemes**

Country	**Maximum working time[1] (minimum daily rest period, where no maximum daily hours)**	**Method of setting threshold[2]**	**Threshold level**	**Specific maximum overtime limits**	**Conditions for use of overtime (procedures, justifications)**	**Enhanced pay rate and/or time off in lieu**
Australia	38 hours per week but not exceeding 152 hours over a 28 day cycle or an average of 38 over the period of an agreed roster cycle.	Legislation	38 hours per week.	None.	No conditions.	50% pay rate for the first two hours and 100% time thereafter, calculated on a daily basis or time off in lieu by the agreement.
Austria	10 hours per day, 50 hours per week (maximum under certain conditions).	Legislation.	8 hours per day, 40 hours per week, which is above average collectively agreed working time.	5 hours per week, and additional 60 hours per year.	No conditions.	50% pay rate or 50% time off in lieu
Belgium	8 hours per day, 40 hours per week.	Legislation and agreements (at sector or company level).	8 hours per day, 40 hours per week.	None.	May only be used on specific grounds - exceptional peaks of work, force majeure, unforeseeable needs. Authorization procedures vary according to reason.	50% pay rate (100% at weekends and public holidays) - may be converted into time off in lieu if provided for by collective agreement.

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Country	**Maximum working time (minimum daily rest period, where no maximum daily hours)**	**Method of setting threshold**	**Threshold level**	**Specific maximum overtime limits**	**Conditions for use of overtime (procedures, justifications)**	**Enhanced pay rate and/or time off in lieu**
Canada	8 hours per day, 48 hours per week (minimum daily rest period of 11 hours).	Legislation and agreements	44 hours per week.	None.	Individual agreement of the employee required for work over 44 hours per week.	50% pay rate or 50% time off in lieu. Paid time off must be taken within three months of the week in which the overtime was earned or, if the employee agrees in writing, it can be taken within 12 months.
Denmark	48 hours per week (minimum daily rest period of 11 hours).	Agreements (at sector or company level).	37 hours per week (industry sector agreement).	12 hours over 4 weeks (industry sector agreement).	Notice period required (industry sector agreement).	Companies with agreement - increased pay rate at 150 200 %, then time off in lieu for overtime hours over a threshold (8 hours in 4 weeks in industry sector agreement). Companies without agreement - mostly time off in lieu.

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Country	Maximum working time (minimum daily rest period, where no maximum daily hours)	Method of setting threshold	Threshold level	Specific maximum overtime limits	Conditions for use of overtime (procedures, justifications)	Enhanced pay rate and/or time off in lieu
Finland	8 hours per day, 40 hours per week.	Legislation or agreement.	40 hours or collectively agreed working time.	138 hours over a 4-month period, 250 hours per year over statutory threshold of 40 hours, raised by 80 hours per year if the 138 hours over a 4-month period is complied with.	Individual agreement of the employee required for work over 40 hours per week.	50% pay rate for the first 2 hours per day, 100% above that. May be converted into time off in lieu by agreement.
France	10 hours per day, 48 hours per week.	Legislation.	35 hours per week.	180 hours per year or set by collective agreement.	No conditions. Permission from authorities required for exceeding annual limits.	Between 35th and 43rd weekly hour - minimum pay rate of 10% (25% without agreement) or time off in lieu by agreement. From 44th hour - 50% pay rate.
Germany	8 hours per day, 48 hours per week.	Agreements (at sector level).	Varies between sectoral agreements.	Varies between sectoral agreements.	Agreement of works council required, except where sectoral agreement includes specific provision.	Increased pay rate and/or time off in lieu, by collective agreement. (Appropriate are 25% on regular working days, and 50% on Sundays and Holidays.)

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Country	**Maximum working time (minimum daily rest period, where no maximum daily hours)**	**Method of setting threshold**	**Threshold level**	**Specific maximum overtime limits**	**Conditions for use of overtime (procedures, justifications)**	**Enhanced pay rate and/or time off in lieu**
Greece*	9 hours per day, 45 hours per week (assuming five-day week).	Legislation.	40 hours.	Work between 41 and 45 hours a week is called 'extra work'. Work exceeding 9 hours a day and/or 45 hours a week is called 'overtime'. Annual limits, varying by sector and region set every six months by Ministry of Labor. Overtime exceeding the lawful limit or for which the aforementioned procedures are not complied with is unlawful overtime.	Over 45 hours per week requires justification, notification of authorities and record-keeping.	Lawful overtime is paid as follows: up to 120 hours a year by augmenting the hourly wage paid by 40%, while for overtime above the 120 hour limit, the augment is 60%. Unlawful overtime is paid with an 80% augment.
Hungary*	12 hours per day, 48 hours per week.	Legislation.	8 hours per day, 40 hours per week.	250 hours per year, may be raised to 300 hours by agreement.	Reasons required, notice to be given, record-keeping compulsory.	50% pay rare (or time off in lieu by agreement), 100% pay rate for work on a holiday (or 50% if time off in lieu granted.)

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Country	Maximum working time (minimum daily rest period, where no maximum daily hours)	Method of setting threshold	Threshold level	Specific maximum overtime limits	Conditions for use of overtime (procedures, justifications)	Enhanced pay rate and/or time off in lieu
Ireland	48 hours per week (minimum daily rest period of 11 hours).	Agreements.	Varies between (mainly company) agreements (average 39 hours).	2 hours per day, 12 hours per week, 240 hours per year, or 36 hours over 4 consecutive week. Limits can be exceeded with permission from the authorities.	No conditions.	25% pay rate (agreements often lay down higher rates).
Italy	48 hours per week averaged over a 4-month period (minimum daily rest period of 11 hours).	Legislation and agreements (at sector level).	40 hours per week.	250 hours per year (may be lower by agreement).	Collective agreement required (sector or company-level).	10% rate (in absence of agreement on higher rate).
Japan	8 hours per day, 48 hours per week.	Legislation.	8 hours per day, 48 hours per week.	15 hours a week, 27 hours in two weeks, 43 hours in 4 weeks, 45 hours in one month, 81 hours in two months, 120 hours in three months, 360 hours in one year.	Any employer that requires workers to work in excess of statutory working hours or on statutory days off must submit a Notification of Agreement on Overtime and Work on Days off to its local Labor Standards Inspection Office.	25% for additional work on a workday, 35% for holiday work, an additional 25% for work late at night (usually defined as 10 PM to 5 AM), an additional 25% for work exceeding 60 hours in a month.

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Country	Maximum working time (minimum daily rest period, where no maximum daily hours)	Method of setting threshold	Threshold level	Specific maximum overtime limits	Conditions for use of overtime (procedures, justifications)	Enhanced pay rate and/or time off in lieu
Luxembourg*	10 hours per day, 48 hours per week.	Legislation.	8 hours per day, 40 hours per week.	None, but overall statutory daily and weekly working time limits (see first column).	Permitted only on specific grounds (e.g. exceptional cases), permission from the authorities required.	40% for all employees (except for upper management), 50% time off in lieu by agreement. This additional hour is exempt from tax and social contributions.
Netherlands	10 hours per day and 50 hours per week with an average of 40 hours per week over a period of 13 weeks.	Legislation and agreements.	Varies between collective agreements (no fixed level).	None, but overall statutory daily, weekly and quarterly working time limits (including ′incidental hours′), which may be extended within limits by agreement (see first column).	Must be ′incidental′ and not ′structural′. Collective agreements often require agreement of works council and/or employees concerned.	Increased pay rate (100%-200%) and/or time off in lieu, by collective agreement.
Norway	13 hours per day or 48 hours per week.	Legislation.	10 hours per day, 25 hours per month, 200 hours per year	200 hours per year (overtime between 200-400 hours per year allowed by individual agreement).	Permitted only on specific non-permanent grounds (e.g. unforeseen events or volume of work). Subject (if possible) to discussion with (elected) staff representatives and (for overtime between 200-400 hours) to agreement with employee.	40% pay rate (usually 50% by agreement, and 100% after 21:00).

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Country	**Maximum working time (minimum daily rest period, where no maximum daily hours)**	**Method of setting threshold**	**Threshold level**	**Specific maximum overtime limits**	**Conditions for use of overtime (procedures, justifications)**	**Enhanced pay rate and/or time off in lieu**
Poland	10 hours per day, 40 hours per week.	Legislation.	8 hours per day, 40 hours per week (over 5-day week).	4 hours per day, 150 hours per year.	Permitted only on specific grounds (e.g. employers′ special needs or rescue operations), monitored by the authorities.	50% pay rate for the first 2 hours, 100% for further hours (and work at night, on Sunday and holidays. May be converted into time off in lieu at request of employee and with employers agreement.
Portugal	8 hours per day, 40 hours per week (up to 10 hours per day and 50 hours per week, by agreement).	Legislation and agreements.	8 hours per day, 40 hours per week (up to 10 hours per day, 60 hours per week by agreement).	2 hours per day, 200 hours per year.	Permitted only on specific grounds (e.g. unscheduled increased workload or force majeure), record-keeping required.	50% pay rate for 1st hour, 75% thereafter that, 100% on rest days and holidays. Plus time off in lieu at 25% of the hours worked.
Slovakia*	48 hours per week (exemption available by collective agreement and permission from the authorities).	Legislation.	40 hours per week over 5- day week (′regular′ working schedule - daily minimum of 3 hours and maximum of 9 hours).	8 hours per week in a period of at most 4 consecutive months, 150 hours per year (excluding certain overtime, such as in the event of disasters). Up to 400 hours in special cases by company-level agreement and with authorities′ permission.	No conditions for up to 150 hours per year.	25% pay rate (higher by company-level agreement).

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Country	Maximum working time (minimum daily rest period, where no maximum daily hours)	Method of setting threshold	Threshold level	Specific maximum overtime limits	Conditions for use of overtime (procedures, justifications)	Enhanced pay rate and/or time off in lieu
Spain	9 hours per day, 40 hours per week.	Legislation.	40 hours per week, which is above average collectively agreed working time.	80 hours per year.	Requires collective agreement or agreement by employee.	Increased pay rate (average 18%) or time off in lieu, by collective agreement.
Sweden*	8 hours per day, 40 hours per week.	Legislation.	40 hours per week, which is above average collectively agreed working time.	48 hours over a period of four weeks or 50 hours over a calendar month, subject to a maximum of 200 hours per calendar year.	Must be justifiable (e.g. special needs, or employers′ requirements) and often subject to agreement (company or workplace-level). Record-keeping compulsory, monitoring by staff representatives.	Increased pay rate (usually 50% to 100%) or time off in lieu, by collective agreement.
UK	48 hours per week (minimum daily rest period of 11 hours).	Agreements (company-level).	Varies between (company-level) agreements.	None, but overall statutory weekly working time limits (from which individuals may ′opt out′).	No conditions.	Increased pay rate or time off in lieu, by agreement.

[1]However it is described (maximum or standard) in the national regulations
[2]Threshold beyond which increased pay rate or time off in lieu for overtime begins, either called ′maximum working time′, or the ′statutory period′, or equivalent to the collectively agreed working hours, depending on the country.
Source: EIRO

MALTA - Overtime Regulations 2012 (hereinafter the Regulations) have been enacted to regulate overtime. The Regulations provide that full-time employees shall only work overtime as required by their employer provided that their average weekly working time, including overtime, does not exceed an average of 48 hours over the applicable reference period in terms of the Organization of Working Time Regulations, 2003 and that the employee can, however, give his/her consent in writing to work more than this weekly average. The payment of overtime is regulated at the rate of one and a half times the normal rate for work carried out in excess of a 40 hour week, averaged over a four-week period or over the shift cycle at the discretion of the employer. The Regulations also allow the employer to introduce schemes to bank hours, i.e., for higher work activity periods to be redeemed against lower activity periods.

A simple characterization of the firm's production technology is given by

$$Q = F(E, h, K, t)$$

where Q is some measure of output, E is the number of workers employed, h is the number of weekly hours per worker (workweek), K is some measure of the nonlabor inputs (we will use 'capital' and 'nonlabor inputs' interchangeably, and t is a linear time trend that reflects technological change. The standard restrictions hold for the production technology:

$$\begin{aligned}
MP_E &= \frac{\partial F}{\partial E} > 0, \frac{\partial^2 F}{\partial E^2} < 0 \\
MP_h &= \frac{\partial F}{\partial h} > 0, \frac{\partial^2 F}{\partial h^2} < 0 \\
MP_K &= \frac{\partial F}{\partial K} > 0, \frac{\partial^2 F}{\partial K^2} < 0 \\
\frac{\partial F}{\partial t} &> 0,
\end{aligned}$$

where the $MP's$ are the marginal productivities with diminishing returns.

For simplicity and convenience it is assumed that all workers work the same number of hours per week, that there is only one type of labor, and that there is no economic distinction between the number or amount of capital/nonlabor inputs used and the utilization rate/machine hours for each unit of the nonlabor inputs. These restrictions can of course be relaxed to accommodate the employment of different occupational groups of workers with differing work schedules as well as an economic distinction between the level of nonlabor input usage and its utilization rate, e.g. machine hours.

On the cost side of production there is an important distinction between the variable cost of labor, i.e. the hourly wage (w) and the quasi-fixed overhead labor costs (B). Quasi-fixed labor costs are labor costs that vary with the number of workers employed but not with hours worked. In other words these nonwage labor costs are incurred for each worker on the payroll regardless of the number of hours they work. Some examples of quasi-fixed labor costs are listed below:

Quasi-fixed labor costs (B)

Hiring and Training Costs:	advertising jobs, interviewing, recruiting orientation, specific-job training
Employee Fringe Benefits:	legally mandated and non-mandated benefits
Administrative Costs:	record keeping costs

The firm's cost equation can be expressed as

$$C = \{(wh + B)(1 - D) \ + [wh^* + B + \lambda w (h - h^*)(D)]\} E + rK, \tag{1}$$

where h^* is a legally or contractually defined standard workweek, e.g. 40 hours, 35 hours, etc., $\lambda \geq 1$ is the overtime premium, r is the rental rate/user cost of capital, and D is an indicator variable defined as $D = 1(h > h^*)$. In the presence of overtime hours $(h - h^* > 0)$, $D = 1$ and the cost equation becomes

$$C = \{[wh^* + B + \lambda w (h - h^*)]\} E + rK \ .$$

In the absence of an overtime premium $(\lambda = 1)$ or when $h < h^*$, the cost equation simplifies to

$$C = (wh + B)(E) \quad + rK. \tag{2}$$

In the case in which the actual work week equals the legally or contractually defined standard work week $(h = h^*)$, the cost equation is simply

$$C = (wh^* + B)(E) \quad + rK. \tag{3}$$

Marginal input costs are obtained from

$$MC_E = \frac{\partial C}{\partial E}$$
$$MC_h = \frac{\partial C}{\partial h}$$
$$MC_K = \frac{\partial C}{\partial K}.$$

The marginal input costs in each overtime hours regime are summarized below.

regime	MC_E	MC_h	MC_K
$h < h^*$	$wh + B$	wE	r
$h > h^*$	$[wh^* + B + \lambda w (h - h^*)]$	λwE	r
$h = h^*$	$wh^* + B$	$-$	r

II. THEORETICAL COBB-DOUGLAS EXAMPLE

To make things concrete in terms of input demand, a simple Cobb-Douglas technology is assumed:

$$Q = Ae^{gt}E^{\alpha}h^{\beta}K^{\gamma}$$

where $A, g > 0, 0 < \beta < \alpha < 1, 0 < \gamma < 1$, and $0 < \alpha + \beta + \gamma < 1$. The parameter g is the percentage growth rate in output arising from neutral technological change. The marginal productivities for the inputs are given by

$$\begin{aligned} MP_E = & \frac{\partial Q}{\partial E} \\ & = \alpha Ae^{gt}E^{\alpha-1}h^{\beta}K^{\gamma} \end{aligned} \tag{4}$$

$$\begin{aligned} MP_h = & \frac{\partial Q}{\partial h} \\ & = \beta Ae^{gt}E^{\alpha}h^{\beta-1}K^{\gamma} \end{aligned} \tag{5}$$

$$\begin{aligned} MP_K = & \frac{\partial Q}{\partial K} \\ & = \gamma Ae^{gt}E^{\alpha}h^{\beta}K^{\gamma-1}. \end{aligned} \tag{6}$$

Later it will be easier to work with the Cobb-Douglas production function expressed in logs:

$$ln(Q) = ln(A) + gt + \alpha ln(E) + \beta ln(h) + \gamma ln(K).$$

There are four distinct economic objectives that can be considered in obtaining the derived demand for inputs: cost minimization, long-run profit maximization, short-run profit maximization, and output maximization subject to a budget constraint.

III. COBB-DOUGLAS COST-MINIMIZATION EXAMPLE - THEORY

Under cost minimization, the employer seeks to produce a given level of output at the lowest possible cost. An example of such an objective is a branch plant in which production orders are exogenously determined by company headquarters. The plant manager's objective is to produce the assigned level of output at the lowest possible cost. The fundamental cost-minimizing conditions are described by

$$\frac{MP_E}{MP_h} = \frac{MC_E}{MC_h} \tag{7}$$

$$\frac{MP_E}{MP_K} = \frac{MC_E}{MC_K}. \tag{8}$$

It is clear that satisfaction of (7) and (8) implies

$$\frac{MP_h}{MP_K} = \frac{MC_h}{MC_K}. \tag{9}$$

The conditional input demand functions corresponding to h, E, and K are derived below for each of the three overtime hours regimes.

$\underline{h < h^*}$

In this regime it is assumed that the work week is less than the standard work week (or that there are no overtime laws or contractual obligations regarding overtime). We first apply the efficiency condition corresponding to (7):

$$\frac{MP_E}{MP_h} = \frac{\alpha A e^{gt} E^{\alpha-1} h^{\beta} K^{\gamma}}{\beta A e^{gt} E^{\alpha} h^{\beta-1} K^{\gamma}} = \frac{\alpha \ \ h}{\beta \ E} = \frac{MC_E}{MC_h} = \frac{wh + B}{wE}.$$

The employment level E cancels out which allows us to solve for the demand for hours per worker (work week):

$$h = \left(\frac{\beta}{\alpha - \beta}\right)\left(\frac{B}{w}\right), \tag{10}$$

or in logs

$$ln(h) = ln\left(\frac{\beta}{\alpha - \beta}\right) + ln\left(\frac{B}{w}\right).$$

Note that the demand for hours per worker does not depend on Q or r. All that matters is the ratio of overhead labor costs to the straight-time hourly wage rate $\left(\frac{B}{w}\right)$. In fact the optimal weekly labor cost per worker is proportional to the quasi-fixed labor costs:

$$\begin{aligned} wh + B &= (w)\left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{B}{w}\right) + B \\ &= \left(\frac{\alpha}{\alpha-\beta}\right)(B)\,. \end{aligned}$$

To schedule a work week less than the standard work week in the presence of overtime requirements, it must be the case that

$$h = \left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{B}{w}\right) < h^*$$

$\Rightarrow$

$$0 < \frac{B}{w} < \left(\frac{\alpha}{\beta} - 1\right) h^*.$$

Employment and nonlabor input demands can be obtained from the efficiency condition corresponding to (8):

$$\frac{MP_E}{MP_K} = \frac{\alpha A e^{gt} E^{\alpha-1} h^{\beta} K^{\gamma}}{\gamma A e^{gt} E^{\alpha} h^{\beta} K^{\gamma-1}} = \frac{\alpha}{\gamma}\frac{K}{E} = \frac{MC_E}{MC_K} = \frac{wh+B}{r} = \left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{B}{r}\right).$$

$\Rightarrow$

$$\frac{\alpha}{\gamma}\frac{K}{E} = \left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{B}{r}\right).$$

We arrive at the optimal capital/employment ratio:

$$\frac{K}{E} = \left(\frac{\gamma}{\alpha-\beta}\right)\left(\frac{B}{r}\right)$$

or

$$K = \left(\frac{\gamma}{\alpha-\beta}\right)\left(\frac{B}{r}\right)(E)\,. \tag{11}$$

In terms of logs, we have

$$ln\,(K) = ln\left(\frac{\gamma}{\alpha-\beta}\right) + ln\left(\frac{B}{r}\right) + ln\,(E)\,. \tag{12}$$

The conditional input demand function for E can now be derived from the production function after substituting (10) and (11) for h and K:

$$Q = Ae^{gt}E^{\alpha}\left[\left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{B}{w}\right)\right]^{\beta}\left[\left(\frac{\gamma}{\alpha-\beta}\right)\left(\frac{B}{r}\right)(E)\right]^{\gamma}.$$

After taking logs, collecting terms, and solving for $ln\,(E)$, we obtain the conditional input demand function for employment (in logs):

$$\begin{aligned} ln\,(E) = & \left(\frac{1}{\alpha+\gamma}\right)[(\beta+\gamma)\,ln\,(\alpha-\beta) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)] \\ & + \left(\frac{\beta}{\alpha+\gamma}\right) ln\left(\frac{w}{r}\right) - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right) ln\left(\frac{B}{r}\right) + \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \end{aligned} \tag{13}$$

The conditional input demand function for K is obtained by substituting (13) for $ln(E)$ in (12) and collecting terms:

$$\begin{aligned} ln\,(K) = & \left(\frac{1}{\alpha+\gamma}\right)[(\beta+\gamma)\,ln\,(\alpha-\beta) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)] + ln\left(\frac{\gamma}{\alpha-\beta}\right) \\ & + \left(\frac{\beta}{\alpha+\gamma}\right) ln\left(\frac{w}{r}\right) + \left(\frac{\alpha-\beta}{\alpha+\gamma}\right) ln\left(\frac{B}{r}\right) + \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \end{aligned} \tag{14}$$

The partial effects of the variables w, B, r, Q, and t on the input demands are derived as follows. Starting with the demand for the work week, we see that

$$\begin{aligned} \frac{\partial ln(h)}{\partial ln(w)} &= -1 < 0 \\ \frac{\partial ln(h)}{\partial ln(B)} &= 1 > 0 \\ \frac{\partial ln(h)}{\partial ln(r)} &= \frac{\partial ln(h)}{\partial ln(Q)} = \frac{\partial ln(h)}{\partial t} = 0. \end{aligned}$$

Next, we consider the conditional input demand for employment:

$$\frac{\partial ln(E)}{\partial ln(w)} = \frac{\beta}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(E)}{\partial ln(B)} = \frac{-(\beta + \gamma)}{\alpha + \gamma} < 0$$
$$\frac{\partial ln(E)}{\partial ln(r)} = \frac{\gamma}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(E)}{\partial ln(Q)} = \frac{1}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(E)}{\partial t} = \frac{-g}{\alpha + \gamma} < 0.$$

In the case of the conditional input demand for capital, we have:

$$\frac{\partial ln(K)}{\partial ln(w)} = \frac{\beta}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(K)}{\partial ln(B)} = \frac{\alpha - \beta}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(K)}{\partial ln(r)} = \frac{-\alpha}{\alpha + \gamma} < 0$$
$$\frac{\partial ln(K)}{\partial ln(Q)} = \frac{1}{\alpha + \gamma} > 0$$
$$\frac{\partial ln(K)}{\partial t} = \frac{-g}{\alpha + \gamma} < 0.$$

We can also examine the partial effects of w, B, r, Q, and t on aggregate/total hours demanded (H) defined by $H = hE$:

$$\frac{\partial ln(H)}{\partial ln(w)} = \frac{\partial ln(h)}{\partial ln(w)} + \frac{\partial ln(E)}{\partial ln(w)}$$
$$= -1 + \frac{\beta}{\alpha + \gamma} < 0$$

$$\frac{\partial ln(H)}{\partial ln(B)} = \frac{\partial ln(h)}{\partial ln(B)} + \frac{\partial ln(E)}{\partial ln(B)}$$
$$= 1 - \frac{(\beta + \gamma)}{\alpha + \gamma}$$
$$= \frac{\alpha - \beta}{\alpha + \gamma} > 0$$

$$\frac{\partial ln(H)}{\partial ln(r)} = \frac{\partial ln(h)}{\partial ln(r)} + \frac{\partial ln(E)}{\partial ln(r)}$$
$$= 0 + \frac{\gamma}{\alpha + \gamma}$$
$$= \frac{\gamma}{\alpha + \gamma} > 0$$

$$\frac{\partial ln(H)}{\partial ln(Q)} = \frac{\partial ln(h)}{\partial ln(Q)} + \frac{\partial ln(E)}{\partial ln(Q)}$$
$$= 0 + \frac{1}{\alpha + \gamma}$$
$$= \frac{1}{\alpha + \gamma} > 0$$

$$\frac{\partial ln(H)}{\partial t} = \frac{\partial ln(h)}{\partial t} + \frac{\partial ln(E)}{\partial t}$$
$$= 0 - \frac{g}{\alpha + \gamma}$$
$$= \frac{-g}{\alpha + \gamma} < 0.$$

The signs of the partial effects of the variables w, B, r, Q, and t on the input demands are summarized in the table below.

$h < h^*$				
	h	E	K	H
w	-	+	+	-
B	+	-	+	+
r	0	+	-	+
Q	0	+	+	+
t	0	-	-	-

Perhaps the positive effect of the wage rate w on employment E may seem counterintuitive. Since output is held constant, an increase in the wage while reducing the demand for the work week is also leading to the substitution of employment and nonlabor inputs for hours per worker. Nevertheless, total hours of labor demanded decline with an increase in the wage rate.

Next we consider conditional input labor demands in the overtime hours regime.

$\underline{h > h^*}$

We apply the efficiency condition corresponding to (7) in the case of overtime hours:

$$\frac{MP_E}{MP_h} = \frac{\alpha}{\beta}\frac{h}{E} = \frac{MC_E}{MC_h} = \frac{[wh^* + B + \lambda w (h - h^*)]}{\lambda w E}.$$

The employment level E cancels out which allows us to solve for the demand for hours per worker (work week):

$$\begin{aligned} h &= \left(\frac{\beta}{\alpha - \beta}\right)\left(\frac{1-\lambda}{\lambda}\right) h^* + \left(\frac{1}{\lambda}\right)\left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{B}{w}\right) \\ &= \left(\frac{1}{\lambda}\right)\left(\frac{\beta}{\alpha-\beta}\right)\left[(1-\lambda) h^* + \left(\frac{B}{w}\right)\right] > h^* > 0. \end{aligned} \tag{15}$$

$\Rightarrow$

$$\frac{B}{w} > \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*.$$

The demand for overtime hours is easily obtained as

$$h - h^* = \left[\left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{1-\lambda}{\lambda}\right) - 1\right] h^* + \left(\frac{1}{\lambda}\right)\left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{B}{w}\right) > 0. \tag{16}$$

As in the case when $h < h^*$, the demand for hours per worker does not depend on Q nor does it depend on r. Upon substitution of (16) for overtime hours in the expression for marginal cost in the overtime regime one obtains

$$MC_E = \left(\frac{\alpha}{\alpha-\beta}\right)(w)\left[(1-\lambda)\,h^* + \frac{B}{w}\right].$$

Employment and nonlabor input demands can be obtained from the efficiency condition corresponding to (8):

$$\frac{MP_E}{MP_K} = \frac{\alpha \;\; K}{\gamma \;\; E} = \frac{MC_E}{MC_K} = \frac{[wh^* + B + \lambda w\,(h-h^*)]}{r} = \left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{w}{r}\right)\left[(1-\lambda)\,h^* + \frac{B}{w}\right].$$

$\Rightarrow$

$$\frac{\alpha \;\; K}{\gamma \;\; E} = \left(\frac{\alpha}{\alpha-\beta}\right)\left(\frac{w}{r}\right)\left[(1-\lambda)\,h^* + \frac{B}{w}\right].$$

We next arrive at the optimal capital/employment ratio:

$$\frac{K}{E} = \left(\frac{\gamma}{\alpha-\beta}\right)\left(\frac{w}{r}\right)\left[(1-\lambda)\,h^* + \frac{B}{w}\right],$$

or

$$K = \left(\frac{\gamma}{\alpha-\beta}\right)\left(\frac{w}{r}\right)\left[(1-\lambda)\,h^* + \frac{B}{w}\right](E). \tag{17}$$

In terms of logs we see that

$$ln\,(K) = ln\left(\frac{\gamma}{\alpha-\beta}\right) + ln\left(\frac{w}{r}\right) + ln\left[(1-\lambda)\,h^* + \frac{B}{w}\right] + ln\,(E). \tag{18}$$

In order to derive expressions for conditional input demands for labor in the overtime regime, it is easier to work in terms of logs. The demand for the workweek in an overtime situation is obtained as

$$ln(h) = -ln(\lambda) \; + ln\left(\frac{\beta}{\alpha-\beta}\right) + ln\left[(1-\lambda)\,h^* + \left(\frac{B}{w}\right)\right]. \tag{19}$$

The conditional input demand function for $ln(E)$ can now be derived from the production function (in logs) after substituting (19) and (21) for $ln(h)$ and $ln(K)$:

$$
\begin{aligned}
ln(Q) &= ln(A) + gt + \alpha ln(E) \\
&\quad + \beta \left\{ -ln(\lambda) \ + ln\left(\frac{\beta}{\alpha-\beta}\right) + ln\left[(1-\lambda)\, h^* + \frac{B}{w}\right] \right\} \\
&\quad + \gamma \ \left\{ ln\left(\frac{\gamma}{\alpha-\beta}\right) + ln\left(\frac{w}{r}\right) + ln\left[(1-\lambda)\, h^* + \frac{B}{w}\right] + ln\,(E) \right\}.
\end{aligned}
$$

After collecting terms and solving for $ln\,(E)$, we obtain the conditional input demand function for employment (in logs):

$$
\begin{aligned}
ln(E) &= \left(\frac{-1}{\alpha+\gamma}\right) \left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda) \right] \\
&\quad - \left(\frac{\gamma}{\alpha+\gamma}\right) ln\left(\frac{w}{r}\right) - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right) ln\left[(1-\lambda)\, h^* + \frac{B}{w}\right] \\
&\quad + \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \qquad (20)
\end{aligned}
$$

The conditional input demand function for K (in logs) is obtained by substituting (20) for $ln(E)$ in (18), and collecting terms:

$$
\begin{aligned}
ln(K) &= \left(\frac{-1}{\alpha+\gamma}\right) \left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda) \right] + ln\left(\frac{\gamma}{\alpha-\beta}\right) \\
&\quad \left(\frac{\alpha}{\alpha+\gamma}\right) ln\left(\frac{w}{r}\right) + \left(\frac{\alpha-\beta}{\alpha+\gamma}\right) ln\left[(1-\lambda)\, h^* + \frac{B}{w}\right] \\
&\quad + \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \qquad (21)
\end{aligned}
$$

The partial effects of the variables w, B, r, Q, t, λ, and h^* on the conditional input demands in an overtime regime are derived below. Starting with the demand for the work week, we see that

$$\frac{\partial ln(h)}{\partial ln(w)} = w\frac{\partial ln(h)}{\partial w} = \left(\frac{-1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right) < 0$$

$$\frac{\partial ln(h)}{\partial ln(B)} = B\frac{\partial ln(h)}{\partial B} = \left(\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right) > 0$$

$$\frac{\partial ln(h)}{\partial ln(r)} = \frac{\partial ln(h)}{\partial ln(Q)} = \frac{\partial ln(h)}{\partial t} = 0$$

$$\frac{\partial ln(h)}{\partial ln(\lambda)} = \lambda\frac{\partial ln(h)}{\partial \lambda} = -\left[1 + \left(\frac{\lambda h^*}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\right] < 0$$

$$\frac{\partial ln(h)}{\partial ln(h^*)} = h^*\frac{\partial ln(h)}{\partial h^*} = \frac{(1-\lambda)\,h^*}{(1-\lambda)\,h^* + \dfrac{B}{w}} < 0$$

$$\frac{\partial (h-h^*)}{\partial h^*} = \left(\frac{\beta}{\alpha-\beta}\right)\left(\frac{1-\lambda}{\lambda}\right) - 1 = \frac{\beta-\alpha\lambda}{(\alpha-\beta)\,(\lambda)} < 0.$$

Next, we consider the conditional input demand for employment:

$$\frac{\partial ln(E)}{\partial ln(w)} = w\frac{\partial ln(E)}{\partial w}$$

$$= \left(\frac{1}{\alpha+\gamma}\right)\left\{\left[\frac{\beta+\gamma}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]\left(\frac{B}{w}\right)-\gamma\right\} > 0$$

$$\frac{\partial ln(E)}{\partial ln(B)} = B\frac{\partial ln(E)}{\partial B}$$

$$= \left(\frac{-1}{\alpha+\gamma}\right)\left[\frac{\beta+\gamma}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]\left(\frac{B}{w}\right) < 0$$

$$\frac{\partial ln(E)}{\partial ln(r)} = \frac{\gamma}{\alpha+\gamma} > 0$$

$$\frac{\partial ln(E)}{\partial ln(Q)} = \frac{1}{\alpha+\gamma} > 0$$

$$\frac{\partial ln(E)}{\partial t} = \frac{-g}{\alpha+\gamma} < 0$$

$$\frac{\partial ln(E)}{\partial ln(\lambda)} = \lambda\frac{\partial ln(E)}{\partial \lambda}$$

$$= \left(\frac{\beta}{\alpha+\gamma}\right)\left\{1+\left(\frac{\beta+\gamma}{\beta}\right)\left[\frac{\lambda h^{*}}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]\right\} > 0$$

$$\frac{\partial ln(E)}{\partial ln(h^{*})} = h^{*}\frac{\partial ln(E)}{\partial h^{*}}$$

$$= \left(\frac{\beta+\gamma}{\alpha+\gamma}\right)\left[\frac{(\lambda-1)\,h^{*}}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right] > 0.$$

In the case of the conditional input demand for capital, we have:

$$\frac{\partial ln(K)}{\partial ln(w)} = w\frac{\partial ln(K)}{\partial w}$$

$$= \left(\frac{\alpha}{\alpha+\gamma}\right)\left\{\left(\frac{\beta-\alpha}{\alpha}\right)\left[\frac{1}{(1-\lambda)\,h^{*}+\frac{B}{w}}\right]+1\right\} > 0$$

$$\frac{\partial ln(K)}{\partial ln(B)} = B\frac{\partial ln(K)}{\partial B}$$

$$= \left(\frac{\alpha-\beta}{\alpha+\gamma}\right)\left[\frac{1}{(1-\lambda)\,h^{*}+\frac{B}{w}}\right] > 0$$

$$\frac{\partial ln(K)}{\partial ln(r)} = \frac{-\alpha}{\alpha+\gamma} < 0$$

$$\frac{\partial ln(K)}{\partial ln(Q)} = \frac{1}{\alpha+\gamma} > 0$$

$$\frac{\partial ln(K)}{\partial t} = \frac{-g}{\alpha+\gamma} < 0$$

$$\frac{\partial ln(K)}{\partial ln(\lambda)} = \lambda\frac{\partial ln(K)}{\partial \lambda}$$

$$= \left(\frac{\beta}{\alpha+\gamma}\right)\left\{\left(\frac{\beta-\alpha}{\beta}\right)\left[\frac{\lambda h^{*}}{(1-\lambda)\,h^{*}+\frac{B}{w}}\right]+1\right\} > 0$$

since $\frac{B}{w} > \left(\frac{\alpha}{\beta}\lambda-1\right)h^{*}$

$$\frac{\partial ln(K)}{\partial ln(h^{*})} = h^{*}\frac{\partial ln(K)}{\partial h^{*}}$$

$$= \left(\frac{\alpha-\beta}{\alpha+\gamma}\right)\left[\frac{(1-\lambda)\,h^{*}}{(1-\lambda)\,h^{*}+\frac{B}{w}}\right] < 0$$

We next examine the partial effects of $w, B, r, Q,\ t, \lambda,$ and h^* on aggregate/total hours demanded ($H = hE$):

$$\frac{\partial ln(H)}{\partial ln(w)} = \frac{\partial ln(h)}{\partial ln(w)} + \frac{\partial ln(E)}{\partial ln(w)}$$

$$= \left(\frac{-1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right) + \left(\frac{1}{\alpha+\gamma}\right)\left\{\left[\frac{\beta+\gamma}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) - \gamma\right\}$$

$$= \left(\frac{1}{\alpha+\gamma}\right)\left\{\left(\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right)(\beta-\alpha) - \gamma\right\} < 0$$

$$\frac{\partial ln(H)}{\partial ln(B)} = \frac{\partial ln(h)}{\partial ln(B)} + \frac{\partial ln(E)}{\partial ln(B)}$$

$$= \left(\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right) - \left(\frac{1}{\alpha+\gamma}\right)\left[\frac{\beta+\gamma}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right)$$

$$= \left(\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right)\left[1 - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right)\right] > 0$$

$$\frac{\partial ln(H)}{\partial ln(r)} = \frac{\partial ln(h)}{\partial ln(r)} + \frac{\partial ln(E)}{\partial ln(r)}$$

$$= 0 + \frac{\gamma}{\alpha+\gamma}$$

$$= \frac{\gamma}{\alpha+\gamma} > 0$$

$$\frac{\partial ln(H)}{\partial ln(Q)} = \frac{\partial ln(h)}{\partial ln(Q)} + \frac{\partial ln(E)}{\partial ln(Q)}$$

$$= 0 + \frac{1}{\alpha+\gamma}$$

$$= \frac{1}{\alpha+\gamma} > 0$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial t} &= \frac{\partial ln(h)}{\partial t} + \frac{\partial ln(E)}{\partial t} \\
&= 0 - \frac{g}{\alpha+\gamma} \\
&= \frac{-g}{\alpha+\gamma} < 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial ln(\lambda)} &= \frac{\partial ln(h)}{\partial ln\left(\lambda\right)} + \frac{\partial ln(E)}{\partial ln\left(\lambda\right)} \\
&= -\left[1 + \left(\frac{\lambda h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}}\right)\right] \\
&+ \left(\frac{\beta}{\alpha+\gamma}\right)\left\{1 + \left(\frac{\beta+\gamma}{\beta}\right)\left[\frac{\lambda h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}}\right]\right\} \\
&= \left(\frac{1}{\alpha+\gamma}\right)\left[\frac{\lambda h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}} + \beta - 1\right] < 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial ln(h^*)} &= \frac{\partial ln(h)}{\partial ln\left(h^*\right)} + \frac{\partial ln(E)}{\partial ln\left(h^*\right)} \\
&= \left[\frac{\left(1-\lambda\right)h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}}\right] - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right)\left[\frac{\left(1-\lambda\right)h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}}\right] \\
&= \left[\frac{\left(1-\lambda\right)h^*}{\left(1-\lambda\right)h^* + \dfrac{B}{w}}\right]\left(\frac{\alpha-\beta}{\alpha+\gamma}\right) < 0.
\end{aligned}
$$

The signs of the partial effects of the variables on input demands are summarized in the table below.

$h > h^*$				
	h	E	K	H
w	-	+	+	-
B	+	-	+	+
r	0	+	-	+
Q	0	+	+	+
t	0	-	-	-
λ	-	+	+	-
h^*	-	+	-	-

The last case to be considered is that of the standard workweek.

$\underline{h = h^*}$

The standard workweek is efficient over a range of values for the ratio B/W. To be precise, for $\left(\frac{\alpha}{\beta} - 1\right) h^* \leq \frac{B}{w} \leq \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*$. This can be illustrated in a graph.

When hours are set at the standard workweek, the production function is given by

$$Q = Ae^{gt}E^{\alpha}\left(h^*\right)^{\beta}K^{\gamma},$$

or in logs by

$$ln(Q) = ln(A) + gt + \alpha ln(E) + \beta ln(h^*) + \gamma ln(K). \tag{22}$$

The corresponding marginal products for the variable inputs are as follows:

$$\begin{aligned} MP_E = & \ \frac{\partial Q}{\partial E} \\ & = \alpha Ae^{gt}E^{\alpha-1}\left(h^*\right)^{\beta}K^{\gamma} \end{aligned} \tag{23}$$

$$\begin{aligned} MP_K = & \ \frac{\partial Q}{\partial K} \\ & = \gamma Ae^{gt}E^{\alpha}\left(h^*\right)^{\beta}K^{\gamma-1}. \end{aligned} \tag{24}$$

The derived conditional input demands for employment and capital are obtained from the cost minimizing efficiency conditions:

$$\frac{MP_E}{MP_K} = \frac{\alpha Ae^{gt}E^{\alpha-1}\left(h^*\right)^{\beta}K^{\gamma}}{\gamma Ae^{gt}E^{\alpha}\left(h^*\right)^{\beta}K^{\gamma-1}} = \frac{\alpha K}{\gamma E} = \frac{MC_E}{MC_K} = \frac{wh^* + B}{r}$$

$\Rightarrow$

$$\frac{\alpha K}{\gamma E} = \frac{wh^* + B}{r}.$$

The optimal capital/employment ratio is therefore given by

$$\frac{K}{E} = \left(\frac{\gamma}{\alpha}\right)\left(\frac{wh^* + B}{r}\right).$$

Solving for K in terms of E, we have

$$K = \left(\frac{\gamma}{\alpha}\right)\left(\frac{wh^* + B}{r}\right)(E),$$

or in logs

$$ln(K) = ln\left(\frac{\gamma}{\alpha}\right) + ln\left(\frac{wh^* + B}{r}\right) + ln(E). \tag{25}$$

The conditional input demand function for employment (in logs) can now be derived from the production function after substituting (25) for $ln(K)$ in (22) and solving for $ln(E)$:

$$ln(Q) = ln(A) + gt + \alpha ln(E) + \beta ln(h^*) + \gamma \left[ln\left(\frac{\gamma}{\alpha}\right) + ln\left(\frac{w}{r}\right) + ln\left(h^* + \frac{B}{w}\right) + ln(E) \right]$$

$\Rightarrow$

$$\begin{aligned} ln(E) = &\left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \gamma ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right] - \left(\frac{\gamma}{\alpha+\gamma}\right) ln\left(\frac{wh^* + B}{r}\right) \\ &+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \end{aligned} \tag{26}$$

In the case of the nonlabor inputs in the standard workweek regime, the conditional input demand function (in logs) is obtained by substituting (26) for $ln(E)$ in (25) and collecting terms:

$$\begin{aligned} ln(K) = &\left(\frac{1}{\alpha+\gamma}\right)\left[\alpha ln\left(\frac{\gamma}{\alpha}\right) - ln(A) - \beta ln(h^*)\right] + \left(\frac{\alpha}{\alpha+\gamma}\right) ln\left(\frac{wh^* + B}{r}\right) \\ &+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q) - \left(\frac{g}{\alpha+\gamma}\right) t. \end{aligned} \tag{27}$$

The partial effects of the variables w, B, r, Q, t, and h^* on the input demands for E and K as well as on aggregate hours (H) are derived as follows. For employment demand we have

$$\begin{aligned}
\frac{\partial ln(E)}{\partial ln(w)} &= w\frac{\partial ln(E)}{\partial w} \\
&= \left(\frac{-\gamma}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) < 0 \\
\frac{\partial ln(E)}{\partial ln(B)} &= B\frac{\partial ln(E)}{\partial B} \\
&= \left(\frac{-\gamma}{\alpha+\gamma}\right)\left(\frac{B}{wh^*+B}\right) < 0 \\
\frac{\partial ln(E)}{\partial ln(r)} &= \frac{\gamma}{\alpha+\gamma} > 0 \\
\frac{\partial ln(E)}{\partial ln(Q)} &= \frac{1}{\alpha+\gamma} > 0 \\
\frac{\partial ln(E)}{\partial t} &= \frac{-g}{\alpha+\gamma} < 0 \\
\frac{\partial ln(E)}{\partial ln(h^*)} &= h^*\frac{\partial ln(E)}{\partial h^*} \\
&= \left(\frac{-\gamma}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) < 0.
\end{aligned}$$

In the case of the nonlabor inputs, the partial effects of $w, B, r, Q,\ t$, and h^* are given by

$$\begin{aligned}
\frac{\partial ln(K)}{\partial ln(w)} &= w\frac{\partial ln(K)}{\partial w} \\
&= \left(\frac{\alpha}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) > 0 \\
\frac{\partial ln(K)}{\partial ln(B)} &= B\frac{\partial ln(K)}{\partial B} \\
&= \left(\frac{\alpha}{\alpha+\gamma}\right)\left(\frac{B}{wh^*+B}\right) > 0 \\
\frac{\partial ln(K)}{\partial ln(r)} &= \frac{-\alpha}{\alpha+\gamma} < 0 \\
\frac{\partial ln(K)}{\partial ln(Q)} &= \frac{1}{\alpha+\gamma} > 0 \\
\frac{\partial ln(K)}{\partial t} &= \frac{-g}{\alpha+\gamma} < 0 \\
\frac{\partial ln(K)}{\partial ln(h^*)} &= h^*\frac{\partial ln(K)}{\partial h^*} \\
&= \left(\frac{\alpha}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) > 0.
\end{aligned}$$

Finally, the partial effects of $w, B, r, Q,$ t, and h^* on aggregate hours demanded $H^* = h^* \text{x} E$ (assuming that the actual work week changes to match the changed new standard workweek) are given by:

$$\begin{aligned}
\frac{\partial ln(H^*)}{\partial ln(w)} &= \frac{\partial ln(E)}{\partial ln(w)} \\
&= \left(\frac{-\gamma}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) < 0 \\
\frac{\partial ln(H^*)}{\partial ln(B)} &= \frac{\partial ln(E)}{\partial ln(B)} \\
&= \left(\frac{-\gamma}{\alpha+\gamma}\right)\left(\frac{B}{wh^*+B}\right) < 0 \\
\frac{\partial ln(H^*)}{\partial ln(r)} &= \frac{\partial ln(E)}{\partial ln(r)} \\
&= \frac{\gamma}{\alpha+\gamma} > 0 \\
\frac{\partial ln(H^*)}{\partial ln(Q)} &= \frac{\partial ln(E)}{\partial ln(Q)} \\
&= \frac{1}{\alpha+\gamma} > 0 \\
\frac{\partial ln(H^*)}{\partial t} &= \frac{\partial ln(E)}{\partial t} \\
&= \frac{-g}{\alpha+\gamma} < 0 \\
\frac{\partial ln(H^*)}{\partial ln(h^*)} &= \frac{\partial ln(h^*)}{\partial ln(h^*)} + \frac{\partial ln(E)}{\partial ln(h^*)} \\
&= 1 - \left(\frac{\gamma}{\alpha+\gamma}\right)\left(\frac{wh^*}{wh^*+B}\right) > 0.
\end{aligned}$$

The signs of the partial effects of the variables on input demands are summarized in the table below.

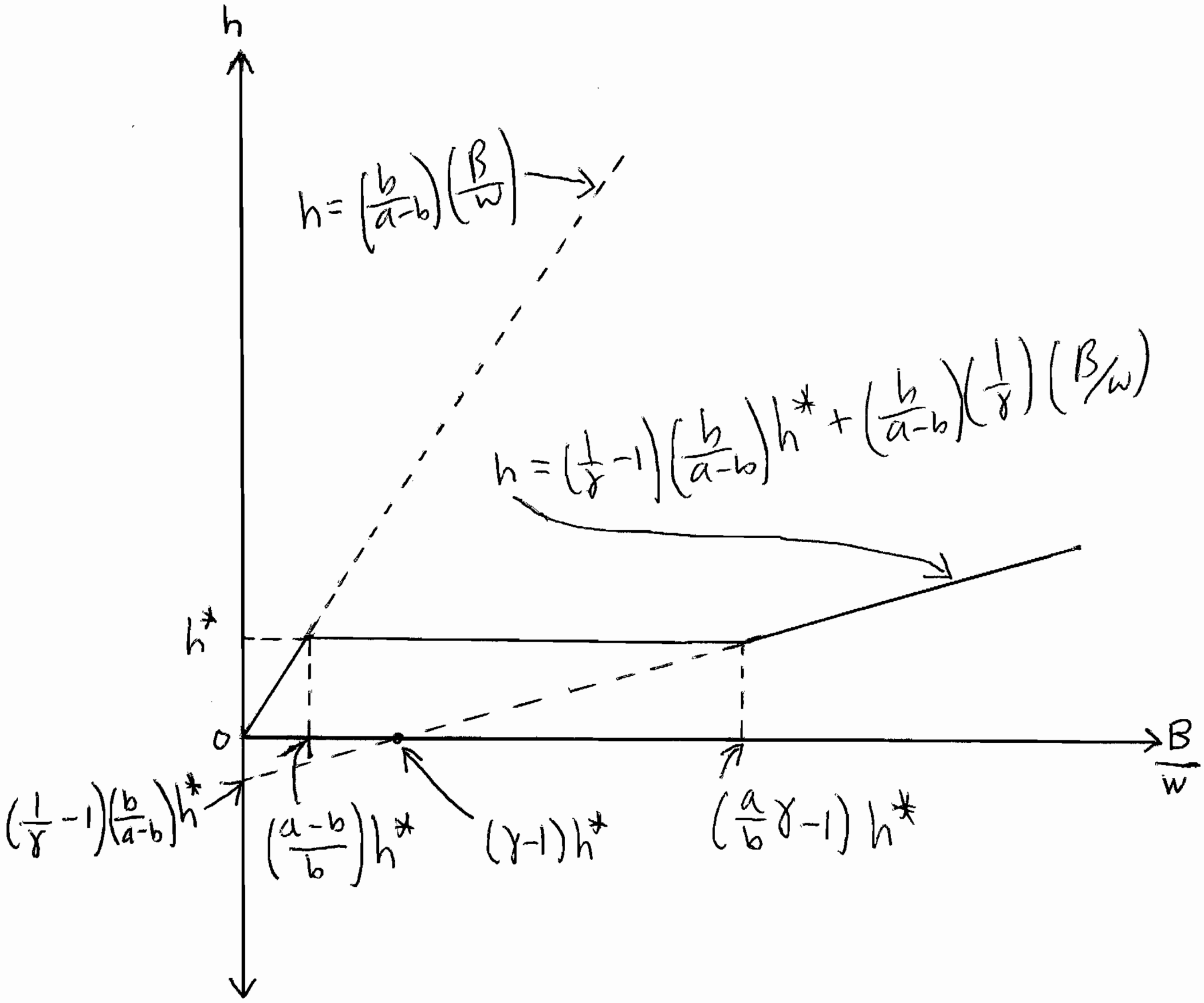
$h = h^*$			
	E	K	H
w	-	+	-
B	-	+	-
r	+	-	+
Q	+	+	+
t	-	-	-
h^*	-	+	+

It should be clear that changes in the ratio B/w can lead to a regime switch. Which regime is cost minimizing is determined according to

$$min\left[C\left(w, B, r, Q, h \mid h < h^*\right), C\left(w, B, r, Q, h \mid h > h^*\right), C\left(w, B, r, Q, h \mid h = h^*\right)\right],$$

which corresponds to

$$\frac{B}{w} < \left(\frac{\alpha}{\beta} - 1\right) h^*$$
$$\frac{B}{w} > \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*$$
$$\left(\frac{\alpha}{\beta} - 1\right) h^* \leq \frac{B}{w} \leq \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*.$$



OVERTIME HOURS

IV. COBB-DOUGLAS COST-MINIMIZATION EXAMPLE - EMPIRICAL

We now consider the empirical estimation of the conditional input demand functions under cost-minimization. We begin with the empirical specification of the conditional input demand functions when there is no overtime. As will be shown below, as long as we have data on w, r, B, and Q it is possible to estimate all of the parameters of the CD technology and the conditional input demand function parameters from data on employment alone. Although the overtime premium λ is treated as a known parameter it could be treated as a variable if it were to change.

$\underline{h < h^*}$

$$ln(h_t) - ln\left(\frac{B_t}{w_t}\right) = a_{01} + \varepsilon_{h1t}$$

$$ln\left(E_t\right) = b_{01} + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) + b_{31}ln(Q_t) + b_{41}t + \varepsilon_{E1t}$$

$$ln\left(K_t\right) = c_{01} + c_{11}ln\left(\frac{w_t}{r_t}\right) + c_{21}ln\left(\frac{B_t}{r_t}\right) + c_{31}ln(Q_t) + c_{41}t + \varepsilon_{K1t}.$$

From the theoretical model we know that there are within and cross-equation restrictions on the parameters:

$$b_{01} = \left(\frac{1}{\alpha+\gamma}\right)\left[(\beta+\gamma)\,ln\left(\alpha-\beta\right) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)\right]$$

$$b_{11} = \frac{\beta}{\alpha+\gamma} > 0$$

$$b_{21} = -\left(\frac{\beta+\gamma}{\alpha+\gamma}\right) < 0$$

$$b_{31} = \frac{1}{\alpha+\gamma} > 0$$

$$b_{41} = \frac{-g}{\alpha+\gamma} < 0$$

$$c_{01} = b_{01} + ln\left(\frac{\gamma}{\alpha - \beta}\right) = b_{01} + ln\left[\frac{-(b_{11} + b_{21})}{1 + b_{21}}\right]$$
$$c_{11} = \frac{\beta}{\alpha + \gamma} = b_{11} > 0$$
$$c_{21} = \frac{\alpha - \beta}{\alpha + \gamma} = 1 + b_{21} > 0$$
$$c_{31} = \frac{1}{\alpha + \gamma} = b_{31} > 0$$
$$c_{41} = \frac{-g}{\alpha + \gamma} = b_{41} < 0$$
$$a_{01} = ln\left(\frac{\beta}{\alpha - \beta}\right) = ln\left(\frac{b_{11}}{1 + b_{21}}\right)$$

The resulting estimating equations incorporating these restrictions are as follows:

$$ln(h_t) - ln\left(\frac{B_t}{w_t}\right) = ln\left(\frac{b_{11}}{1 + b_{21}}\right) + \varepsilon_{h1t} \tag{28}$$

$$ln\,(E_t) = b_{01} + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) + b_{31}ln(Q_t) + b_{41}t + \varepsilon_{E1t} \tag{29}$$

$$ln\,(K_t) - ln\left(\frac{B_t}{r_t}\right) = b_{01} + ln\left[\frac{-(b_{11} + b_{21})}{1 + b_{21}}\right] + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) + b_{31}ln(Q_t) + b_{41}t + \varepsilon_{K1t}. \tag{30}$$

The model could be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions.

The parameters of the underlying CD technology are identified and can be recovered from the estimated conditional input demand function parameters:

$$\tilde{A} = exp\,\left\{\left(\frac{1}{\hat{b}_{31}}\right)\left[\left(\hat{b}_{11} + \hat{b}_{21}\right) ln\left[-\left(\hat{b}_{11} + \hat{b}_{21}\right)\right] - \hat{b}_{21}ln\left(1 + \hat{b}_{21}\right) - \hat{b}_{11}ln\left(\hat{b}_{11}\right) - \hat{b}_{01}\right]\right\}$$
$$\tilde{\alpha} = \frac{1 + \hat{b}_{11} + \hat{b}_{21}}{\hat{b}_{31}}$$
$$\tilde{\beta} = \frac{\hat{b}_{11}}{\hat{b}_{31}}$$
$$\tilde{\gamma} = \frac{-\left(\hat{b}_{11} + \hat{b}_{21}\right)}{\hat{b}_{31}}$$
$$\tilde{g} = \frac{-\hat{b}_{41}}{\hat{b}_{31}}.$$

It is evident that all of the parameters of the conditional input demand functions as well as of the CD production function can be estimated from the demand for employment equation alone (29). While not fully efficient, this strategy would be welcome if data on h_t and K_t were unavailable.

An alternative estimation strategy is to directly estimate the CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$
\begin{aligned}
ln(h_t) - ln\left(\frac{B_t}{w_t}\right) &= ln\left(\frac{\beta}{\alpha-\beta}\right) + \varepsilon_{h1t} \\
ln\left(E_t\right) &= \left(\frac{1}{\alpha+\gamma}\right)\left[\left(\beta+\gamma\right) ln\left(\alpha-\beta\right) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)\right] \\
&+ \left(\frac{\beta}{\alpha+\gamma}\right) ln\left(\frac{w_t}{r_t}\right) - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right) ln\left(\frac{B_t}{r_t}\right) \\
&+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t \; + \varepsilon_{E1t} \\
ln(K_t) &= \left(\frac{1}{\alpha+\gamma}\right)\left[\left(\beta+\gamma\right) ln\left(\alpha-\beta\right) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)\right] \\
&+ ln\left(\frac{\gamma}{\alpha-\beta}\right) + \left(\frac{\beta}{\alpha+\gamma}\right) ln\left(\frac{w_t}{r_t}\right) + \left(\frac{\alpha-\beta}{\alpha+\gamma}\right) ln\left(\frac{B_t}{r_t}\right) \\
&+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t + \varepsilon_{K1t}.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}_{01} &= \left(\frac{1}{\hat{\alpha}+\hat{\gamma}}\right)\left[\left(\hat{\beta}+\hat{\gamma}\right) ln\left(\hat{\alpha}-\hat{\beta}\right) - ln(\hat{A}) - \beta ln(\hat{\beta}) - \hat{\gamma} ln(\hat{\gamma})\right] \\
\tilde{b}_{11} &= \frac{\hat{\beta}}{\hat{\alpha}+\hat{\gamma}} \\
\tilde{b}_{21} &= -\left(\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}}\right) \\
\tilde{b}_{31} &= \frac{1}{\hat{\alpha}+\hat{\gamma}} \\
\tilde{b}_{41} &= \frac{-\hat{g}}{\hat{\alpha}+\hat{\gamma}}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{c}_{01} &= \tilde{b}_{01} + ln\left[\frac{-(\tilde{b}_{11} + \tilde{b}_{21})}{1 + \tilde{b}_{21}}\right] \\
\tilde{c}_{11} &= \tilde{b}_{11} \\
\tilde{c}_{21} &= \ 1 + \tilde{b}_{21} \\
\tilde{c}_{31} &= \ \tilde{b}_{31} \\
\tilde{c}_{41} &= \tilde{b}_{41} \\
\tilde{a}_{01} &= ln\left(\frac{\tilde{b}_{11}}{1 + \tilde{b}_{21}}\right).
\end{aligned}
$$

Next, we consider estimation of the conditional input labor demands in the overtime hours regime.

$\underline{h > h^*}$

$$
ln(h_t) - ln\left[(1-\lambda)\, h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = a_{02} + \ \varepsilon_{h2t}
$$

$$
\begin{aligned}
ln(E_t) &= b_{02} + b_{12} ln\left(\frac{w_t}{r_t}\right) + b_{22} ln\left[(1-\lambda)\, h^* + \frac{B_t}{w_t}\right] \\
&\quad + b_{32} ln(Q_t) + b_{42} t + \varepsilon_{E2t} \\
ln(K_t) &= c_{02} + c_{12} ln\left(\frac{w_t}{r_t}\right) + c_{22} ln\left[(1-\lambda)\, h^* + \frac{B_t}{w_t}\right] \\
&\quad + c_{32} ln(Q_t) + c_{42} t + \varepsilon_{K2t}.
\end{aligned}
$$

From the theoretical model there are again within and cross-equation restrictions on the parameters:

$$
\begin{aligned}
b_{02} &= \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda)\right] \\
b_{12} &= \frac{-\gamma}{\alpha+\gamma} < 0 \\
b_{22} &= -\left(\frac{\beta+\gamma}{\alpha+\gamma}\right) < 0 \\
b_{32} &= \frac{1}{\alpha+\gamma} > 0 \\
b_{42} &= \frac{-g}{\alpha+\gamma} < 0 \\
c_{02} &= b_{02} + ln\left(\frac{\gamma}{\alpha-\beta}\right) = b_{02} + \ ln(-b_{12}) \\
c_{12} &= \frac{\alpha}{\alpha+\gamma} = 1 + b_{12} \ > 0 \\
c_{22} &= \ \frac{\alpha-\beta}{\alpha+\gamma} = 1 + b_{22} \ > 0 \\
c_{32} &= \frac{1}{\alpha+\gamma} = b_{32} > 0 \\
c_{42} &= \frac{-g}{\alpha+\gamma} = b_{42} < 0 \\
a_{02} &= ln\left(\frac{\beta}{\alpha-\beta}\right) = ln\left(b_{12} - b_{22}\right).
\end{aligned}
$$

The resulting estimating equations incorporating the above restrictions are as follows:

$$
ln(h_t) - ln\left[\left(1-\lambda\right)h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = \ ln\left(b_{12} - b_{22}\right) + \ \varepsilon_{h2t} \tag{31}
$$

$$
ln\left(E_t\right) = b_{02} + b_{12}ln\left(\frac{w_t}{r_t}\right) + b_{22}ln\left[\left(1-\lambda\right)h^* + \frac{B_t}{w_t}\right] \ + b_{32}ln(Q_t) + b_{42}t + \varepsilon_{E2t} \tag{32}
$$

$$
\begin{aligned}
ln\left(K_t\right) - ln\left(\frac{w_t}{r_t}\right) - ln\left[\left(1-\lambda\right)h^* + \frac{B}{w}\right] &= b_{02} + \ ln(-b_{12}) \ + b_{12}ln\left(\frac{w_t}{r_t}\right) \\
&\quad + b_{22} \ ln\left[\left(1-\lambda\right)h^* + \frac{B_t}{w_t}\right] \\
&\quad + b_{32}ln(Q_t) + b_{42}t + \varepsilon_{K2t}.
\end{aligned} \tag{33}
$$

The model could be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions.

The parameters of the underlying CD technology are identified and can be recovered from the estimated conditional input demand function parameters:

$$\tilde{A} = exp\left[\left(\hat{b}_{12} - \hat{b}_{22}\right) ln(\lambda) - \left(\hat{b}_{12} - \hat{b}_{22}\right) ln\left(\frac{\hat{b}_{12} - \hat{b}_{22}}{1 + \hat{b}_{22}}\right) + \hat{b}_{12} ln\left(\frac{-\hat{b}_{12}}{1 + \hat{b}_{22}}\right) - \hat{b}_{02}\right]$$

$$\tilde{\alpha} = \frac{1 + \hat{b}_{12}}{\hat{b}_{32}}$$

$$\tilde{\beta} = \frac{\hat{b}_{12} - \hat{b}_{22}}{\hat{b}_{32}}$$

$$\tilde{\gamma} = \frac{-\hat{b}_{12}}{\hat{b}_{32}}$$

$$\tilde{g} = \frac{-\hat{b}_{42}}{\hat{b}_{32}}.$$

All of the parameters of the conditional input demand functions as well as of the CD production function could be estimated from the demand for employment equation alone (32). While not fully efficient, this strategy would be feasible if data on h_t and K_t were unavailable.

An alternative estimation strategy is to directly estimate the CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$ln(h_t) - ln\left[(1-\lambda)\, h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = ln\left(\frac{\beta}{\alpha - \beta}\right) + \varepsilon_{h2t}$$

$$\begin{aligned} ln(E_t) = &\left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda)\right] \\ &- \left(\frac{\gamma}{\alpha+\gamma}\right) ln\left(\frac{w_t}{r_t}\right) - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right) ln\left[(1-\lambda)\, h^* + \frac{B_t}{w_t}\right] \\ &+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t + \varepsilon_{E2t} \end{aligned}$$

$$
\begin{aligned}
ln(K_t) = &\left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A)+\beta ln\left(\frac{\beta}{\alpha-\beta}\right)+\gamma ln\left(\frac{\gamma}{\alpha-\beta}\right)-\beta ln(\lambda)\right]+ln\left(\frac{\gamma}{\alpha-\beta}\right)\\
&+\left(\frac{\alpha}{\alpha+\gamma}\right)ln\left(\frac{w_t}{r_t}\right)+\left(\frac{\alpha-\beta}{\alpha+\gamma}\right)ln\left[(1-\lambda)\,h^*+\frac{B_t}{w_t}\right]\\
&+\frac{1}{\alpha+\gamma}ln(Q_t)-\left(\frac{g}{\alpha+\gamma}\right)t+\varepsilon_{K2t}.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}_{02} &= \left(\frac{1}{\hat{\alpha}+\hat{\gamma}}\right)\left(\frac{-1}{\hat{\alpha}+\hat{\gamma}}\right)\left[ln(\hat{A})+\hat{\beta}ln\left(\frac{\hat{\beta}}{\hat{\alpha}-\hat{\beta}}\right)+\gamma ln\left(\frac{\gamma}{\hat{\alpha}-\hat{\beta}}\right)-\hat{\beta}ln(\lambda)\right]\\
\tilde{b}_{12} &= \frac{-\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}}\\
\tilde{b}_{22} &= -\left(\frac{\hat{\beta}+\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}}\right)\\
\tilde{b}_{32} &= \frac{1}{\hat{\alpha}+\hat{\gamma}}\\
\tilde{b}_{42} &= \frac{-\hat{g}}{\hat{\alpha}+\hat{\gamma}}\\
\tilde{c}_{02} &= \tilde{b}_{02}+\ ln(-\tilde{b}_{12})\\
\tilde{c}_{12} &= 1+\tilde{b}_{12}\\
\tilde{c}_{22} &= 1+\tilde{b}_{22}\\
\tilde{c}_{32} &= \tilde{b}_{32}\\
\tilde{c}_{42} &= \tilde{b}_{42}\\
\tilde{a}_{02} &= ln\left(\tilde{b}_{12}-\tilde{b}_{22}\right).
\end{aligned}
$$

We last consider the estimation of the conditional input demand functions corresponding the standard workweek.

<u>$h = h^*$</u>

$$
\begin{aligned}
ln(E_t) &= b_{03}+b_{13}ln\left(\frac{w_t h^*+B_t}{r_t}\right)+b_{23}\ ln(Q_t)+b_{33}t+\varepsilon_{E3t}\\
ln(K_t) &= c_{03}+c_{13}ln\left(\frac{w_t h^*+B_t}{r_t}\right)+c_{23}\ ln(Q_t)+c_{33}t+\varepsilon_{K3t}
\end{aligned}
$$

From the theoretical model there are again within and cross-equation restrictions on the parameters:

$$
\begin{aligned}
b_{03} &= \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A)+\gamma ln\left(\frac{\gamma}{\alpha}\right)+\beta ln(h^*)\right] = \left(\frac{-1}{\alpha+\gamma}\right)\left[\phi+\gamma ln\left(\frac{\gamma}{\alpha}\right)\right],\\
&\text{where } \phi = ln(A)+\beta ln(h^*)\\
b_{13} &= \frac{-\gamma}{\alpha+\gamma} < 0\\
b_{23} &= \frac{1}{\alpha+\gamma} > 0\\
b_{33} &= \frac{-g}{\alpha+\gamma} < 0\\
c_{03} &= b_{03} + ln\left(\frac{\gamma}{\alpha}\right) = b_{03} + ln\left(\frac{-b_{13}}{1+b_{13}}\right)\\
c_{13} &= \frac{\alpha}{\alpha+\gamma} = 1+b_{13} \quad > 0\\
c_{23} &= \frac{1}{\alpha+\gamma} = b_{23} \ > 0\\
c_{33} &= \frac{-g}{\alpha+\gamma} = b_{33} < 0.
\end{aligned}
$$

The resulting estimating equations incorporating the above restrictions are as follows:

$$ln(E_t) = b_{03} + b_{13} ln\left(\frac{w_t h^* + B_t}{r_t}\right) + b_{23}\ ln(Q_t) + b_{33}t + \varepsilon_{E3t} \tag{34}$$

$$
\begin{aligned}
ln(K_t) - ln\left(\frac{w_t h^* + B_t}{r_t}\right) &= b_{03} + ln\left(\frac{-b_{13}}{1+b_{13}}\right) + b_{13} ln\left(\frac{w_t h^* + B_t}{r_t}\right)\\
&\quad + b_{23}\ ln(Q_t) + b_{33}t + \varepsilon_{K3t}.
\end{aligned} \tag{35}
$$

As in the other hours regimes, the model could be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions.

The following parameters of the underlying CD technology are identified and can be recovered from the estimated conditional input demand function parameters:

$$\tilde{\alpha} = \frac{1+\hat{b}_{13}}{\hat{b}_{23}}$$
$$\tilde{\gamma} = \frac{-\hat{b}_{13}}{\hat{b}_{23}}$$
$$\tilde{g} = \frac{-\hat{b}_{33}}{\hat{b}_{23}}$$
$$\tilde{\phi} = \left(\frac{-1}{\hat{b}_{23}}\right)\left[\hat{b}_{03} + \hat{b}_{13} ln\left(\frac{-\hat{b}_{13}}{1+\hat{b}_{13}}\right)\right].$$

Perhaps not surprisingly, the CD production function parameters A and β are not identified from data on just the invariant standard work week. One would need variation in h^* to identify these parameters. Without such variation, all that can be identified is $\phi = ln(A) + \beta ln(h^*)$.

All of the identified parameters of the conditional input demand functions as well as of the CD production function could be estimated from the demand for employment equation alone (34). Although not fully efficient, this strategy would be feasible if data on K_t were unavailable.

An alternative estimation strategy is to directly estimate the identified CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$ln(E_t) = \left(\frac{-1}{\alpha+\gamma}\right)\left[\phi + \gamma ln\left(\frac{\gamma}{\alpha}\right)\right] - \left(\frac{\gamma}{\alpha+\gamma}\right) ln\left(\frac{w_t h^* + B_t}{r_t}\right)$$
$$+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t + \varepsilon_{E3t}.$$

$$ln(K_t) = \left(\frac{1}{\alpha+\gamma}\right)\left[\alpha ln\left(\frac{\gamma}{\alpha}\right) - \phi\right] + \left(\frac{\alpha}{\alpha+\gamma}\right) ln\left(\frac{w_t h^* + B_t}{r_t}\right)$$
$$+ \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t + \varepsilon_{K3t}.$$

$$\tilde{b}_{03} = \left(\frac{-1}{\hat{\alpha}+\hat{\gamma}}\right)\left[\hat{\phi}+\hat{\gamma}ln\left(\frac{\hat{\gamma}}{\hat{\alpha}}\right)\right]$$
$$\tilde{b}_{13} = \frac{-\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}}$$
$$\tilde{b}_{23} = \frac{1}{\hat{\alpha}+\hat{\gamma}}$$
$$\tilde{b}_{33} = \frac{-\hat{g}}{\hat{\alpha}+\hat{\gamma}}$$
$$\tilde{c}_{03} = \tilde{b}_{03} + ln\left(\frac{-\tilde{b}_{13}}{1+\tilde{b}_{13}}\right)$$
$$\tilde{c}_{13} = 1 + \tilde{b}_{13}$$
$$\tilde{c}_{23} = \tilde{b}_{23}$$
$$\tilde{c}_{33} = \tilde{b}_{33}.$$

If the available data spans all three hours regimes and for each observation it is known which regime is in effect, then all of the conditional input demand functions could be jointly estimated with cross-equation restrictions.

Joint Estimation Across Hours Regimes

We note below the cross-equation, conditional input demand function parameter restrictions across hours regimes.

$$\begin{aligned} b_{02} &= \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A)+\beta ln\left(\frac{\beta}{\alpha-\beta}\right)+\gamma ln\left(\frac{\gamma}{\alpha-\beta}\right)-\beta ln(\lambda)\right] \\ &= b_{01} - b_{11}ln(\lambda) \end{aligned}$$

$$b_{12} = \frac{-\gamma}{\alpha+\gamma} = b_{11} + b_{21}$$
$$b_{22} = -\left(\frac{\beta+\gamma}{\alpha+\gamma}\right) = b_{21}$$
$$b_{32} = \frac{1}{\alpha+\gamma} = b_{31}$$
$$b_{42} = \frac{-g}{\alpha+\gamma} = b_{41}$$

$$
\begin{aligned}
c_{02} &= b_{02} + \ ln(-b_{12}) \\
&= b_{01} - b_{11} ln(\lambda) + ln\left[-\left(b_{11} + b_{21}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{12} &= 1 + b_{12} \\
&= 1 + b_{11} + b_{21}
\end{aligned}
$$

$$
\begin{aligned}
c_{22} &= \ 1 + b_{22} \\
&= 1 + b_{21}
\end{aligned}
$$

$$
\begin{aligned}
c_{32} &= b_{32} = b_{31} \\
c_{42} &= b_{42} = b_{41}
\end{aligned}
$$

$$
\begin{aligned}
a_{02} &= ln\left(b_{12} - b_{22}\right) \\
&= \ ln\left(b_{11}\right)
\end{aligned}
$$

$$
\begin{aligned}
b_{03} = & \ \left(\frac{-1}{\alpha + \gamma}\right)\left[ln(A) + \gamma ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right] \\
&= b_{01} + b_{21} ln\left(1 + b_{21}\right) \ + b_{11} ln\left(\frac{b_{11}}{h^*}\right) - \left(b_{11} + b_{21}\right) ln\left(1 + b_{11} + b_{21}\right)
\end{aligned}
$$

$$
\begin{aligned}
b_{13} &= \frac{-\gamma}{\alpha + \gamma} = b_{11} + b_{21} \\
b_{23} &= \ \frac{1}{\alpha + \gamma} = b_{31} \\
b_{33} &= \frac{-g}{\alpha + \gamma} = b_{41}
\end{aligned}
$$

$$
\begin{aligned}
c_{03} &= b_{03} + ln\left(\frac{-b_{13}}{1+b_{13}}\right) \\
&= b_{01} + b_{21} ln\left(1+b_{21}\right) \ + b_{11} ln\left(\frac{b_{11}}{h^*}\right) - \left(b_{11}+b_{21}\right) ln\left(1+b_{11}+b_{21}\right) \\
&+ ln\left(\frac{- \ \left(b_{11}+b_{21}\right)}{1+b_{11}+b_{21}}\right) \\
&= b_{01} + b_{21} ln\left(1+b_{21}\right) \ + b_{11} ln\left(\frac{b_{11}}{h^*}\right) - \left(1+b_{11}+b_{21}\right) ln\left(1+b_{11}+b_{21}\right) \\
&+ ln\left[-\left(b_{11}+b_{21}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
c_{13} &= 1 + b_{13} \\
&= 1 + b_{11} + b_{21}
\end{aligned}
$$

$$
\begin{aligned}
c_{23} &= \ b_{23} = b_{31} \\
c_{33} &= b_{33} = b_{41}
\end{aligned}
$$

We define indicator variables for each hours regime as follows: $D_{1t} = 1(h_t < h^*), D_{2t} = 1(h_t > h^*)$, and $D_{3t} = 1 - D_{1t} - D_{2t} = 1(h_t = h^*)$. The demand system for the inputs is specified below.

$$
\begin{aligned}
ln(h_t) = D_{1t} &\left[ln\left(\frac{b_{11}}{1+b_{21}}\right) + ln\left(\frac{B_t}{w_t}\right)\right] \\
+ \ D_{2t} &\left\{ln\left(b_{11}\right) + ln\left[\left(1-\lambda\right)h^* + \left(\frac{B_t}{w_t}\right)\right] - ln(\lambda)\right\} \\
+ D_{3t} & ln(h^*) + \varepsilon_{ht}
\end{aligned}
$$

$$
\begin{aligned}
ln\left(E_t\right) = D_{1t} & \left[b_{01} + b_{11} ln\left(\frac{w_t}{r_t}\right) + b_{21} ln\left(\frac{B_t}{r_t}\right) + b_{31} ln(Q_t) + b_{41} t\right] \\
& + D_{2t}\left\{b_{01} - b_{11} ln(\lambda) + \left(b_{11}+b_{21}\right) ln\left(\frac{w_t}{r_t}\right) + b_{21} ln\left[\left(1-\lambda\right) h^{*} + \frac{B_t}{w_t}\right]\right. \\
& \left. + b_{31} ln(Q_t) + b_{41} t\right\} \\
& + D_{3t}\left[b_{01} + b_{21} ln\left(1+b_{21}\right) + b_{11} ln\left(\frac{b_{11}}{h^{*}}\right) - \left(b_{11}+b_{21}\right) ln\left(1+b_{11}+b_{21}\right)\right. \\
& \left. + \left(b_{11}+b_{21}\right) ln\left(\frac{w_t h^{*} + B_t}{r_t}\right) + b_{31}\ ln(Q_t) + b_{41} t\right] + \varepsilon_{Et}
\end{aligned}
$$

$$
\begin{aligned}
ln\left(K_t\right) \ = D_{1t} & \left\{b_{01} + ln\left[\frac{-\left(b_{11}+b_{21}\right)}{1+b_{21}}\right] + b_{11} ln\left(\frac{w_t}{r_t}\right) + \left(1+b_{21}\right) ln\left(\frac{B_t}{r_t}\right)\right. \\
& \left. + b_{31} ln(Q_t) + b_{41} t\right\} \\
& + D_{2t}\ \left\{b_{01} - b_{11} ln(\lambda) \ + \left(1+b_{11}+b_{21}\right) ln\left(\frac{w_t}{r_t}\right)\right. \\
& \left. + \left(1+b_{21}\right)\ ln\left[\left(1-\lambda\right) h^{*} + \frac{B_t}{w_t}\right] + b_{31} ln(Q_t) + b_{41} t\right\} + \varepsilon_{Kt} \\
& + D_{3t}\left\{b_{01} + b_{21} ln\left(1+b_{21}\right) + b_{11} ln\left(\frac{b_{11}}{h^{*}}\right) - \left(1+b_{11}+b_{21}\right) ln\left(1+b_{11}+b_{21}\right)\right. \\
& + ln\left[-\left(b_{11}+b_{21}\right)\right] + \left(1+b_{11}+b_{21}\right) ln\left(\frac{w_t h^{*} + B_t}{r_t}\right) \\
& \left. + b_{31}\ ln(Q_t) + b_{41} t\right\} + \varepsilon_{Kt}.
\end{aligned}
$$

As before, another strategy is to jointly estimate the conditional input demand system through direct estimation of the CD production function parameters.

$$
\begin{aligned}
ln(h_t) = D_{1t} & \left[ln\left(\frac{\beta}{\alpha-\beta}\right) + ln\left(\frac{B_t}{w_t}\right)\right] \\
& + \ D_{2t}\left\{ln\left(\frac{\beta}{\alpha-\beta}\right) + ln\left[\left(1-\lambda\right) h^{*} + \left(\frac{B_t}{w_t}\right)\right] - ln(\lambda)\right\} \\
& + D_{3t} ln(h^{*}) + \varepsilon_{ht}
\end{aligned}
$$

$$
\begin{aligned}
ln\left(E_t\right) = D_{1t} \Bigg\{ & \left(\frac{1}{\alpha+\gamma}\right)\left[\left(\beta+\gamma\right) ln\left(\alpha-\beta\right) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)\right] \\
& + \frac{\beta}{\alpha+\gamma} ln\left(\frac{w_t}{r_t}\right) - \frac{\beta+\gamma}{\alpha+\gamma} ln\left(\frac{B_t}{r_t}\right) \\
& + \frac{1}{\alpha+\gamma} ln(Q_t) - \frac{g}{\alpha+\gamma} t \Bigg\} \\
& + D_{2t} \Bigg\{ \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda)\right] \\
& - \frac{\gamma}{\alpha+\gamma} ln\left(\frac{w_t}{r_t}\right) - \left(\frac{\beta+\gamma}{\alpha+\gamma}\right) ln\left[\left(1-\lambda\right) h^* + \frac{B_t}{w_t}\right] \\
& + \frac{1}{\alpha+\gamma} ln(Q_t) - \frac{g}{\alpha+\gamma} t \Bigg\} \\
& + D_{3t} \Bigg\{ \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \beta ln(h^*) + \gamma ln\left(\frac{\gamma}{\alpha}\right)\right] - \left(\frac{\gamma}{\alpha+\gamma}\right) ln\left(\frac{w_t h^* + B_t}{r_t}\right) \\
& + \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t \Bigg\} + \varepsilon_{Et}
\end{aligned}
$$

$$
\begin{aligned}
ln(K_t) = D_{1t} \Bigg\{ & \left(\frac{1}{\alpha+\gamma}\right)\left[\left(\beta+\gamma\right) ln\left(\alpha-\beta\right) - ln(A) - \beta ln(\beta) - \gamma ln(\gamma)\right] \\
& + ln\left(\frac{\gamma}{\alpha-\beta}\right) + \frac{\beta}{\alpha+\gamma} ln\left(\frac{w_t}{r_t}\right) + \left(\frac{\alpha-\beta}{\alpha+\gamma}\right) ln\left(\frac{B_t}{r_t}\right) \\
& + \frac{1}{\alpha+\gamma} ln(Q_t) - \frac{g}{\alpha+\gamma} t \Bigg\} \\
& + D_{2t} \Bigg\{ \left(\frac{-1}{\alpha+\gamma}\right)\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - \beta ln(\lambda)\right] + ln\left(\frac{\gamma}{\alpha-\beta}\right) \\
& + \frac{\alpha}{\alpha+\gamma} ln\left(\frac{w_t}{r_t}\right) + \left(\frac{\alpha-\beta}{\alpha+\gamma}\right) ln\left[\left(1-\lambda\right) h^* + \frac{B_t}{w_t}\right] \\
& + \frac{1}{\alpha+\gamma} ln(Q_t) - \frac{g}{\alpha+\gamma} t \Bigg\} \\
& + D_{3t} \Bigg\{ \left(\frac{1}{\alpha+\gamma}\right)\left[\alpha ln\left(\frac{\gamma}{\alpha}\right) - ln(A) - \beta ln(h^*)\right] + \left(\frac{\alpha}{\alpha+\gamma}\right) ln\left(\frac{w_t h^* + B_t}{r_t}\right) \\
& + \left(\frac{1}{\alpha+\gamma}\right) ln(Q_t) - \left(\frac{g}{\alpha+\gamma}\right) t \Bigg\} + \varepsilon_{Kt}.
\end{aligned}
$$

As long as one knows which regime corresponds to each period in the data, the CD production function parameters and the conditional input demand function parameters can be estimated from data on employment alone. Furthermore, if w_t, r_t, B_t, and Q_t are exogenous, the NLSUR estimator is consistent. For example, these variables could plausibly be assumed to be exogenous for a branch plant of a multi-plant firm. If Q_t were endogenous, then a nonlinear 3SLS estimator would be consistent.