Demand for Employment and Hours Under Long-run Profit Max: notes

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Abstract

These notes extend the Cobb-Douglas example of employment and overtime hours to long-run profit maximization.

I. COBB-DOUGLAS LONG-RUN PROFIT MAXIMIZATION - THEORY

Under long-run profit maximization, the firm seeks to employ the level of inputs that maximize profits when all inputs are variable. This is of course equivalent to producing the level of output that maximizes profits. For convenience and simplicity, it is assumed that output price is exogenous. This assumption means that price (p) and marginal revenue (MR) are the same.

As we have seen in the cost-minimization example, the demand for hours per worker depends only on the input prices w and B. In particular that meant that the demand for h did not depend on Q or r. In the context of profit maximization, Q is endogenous and the demand for hours per worker does not depend on p or r. Since profit maximization implies cost-minimization, we already know the optimal capital/employment ratio for each hours regime. Therefore, in order to obtain the demand functions for employment (E) and (K) it is necessary to solve for their optimal values from the single profit maximization condition given by

$$p \cdot MP_E = MC_E,$$

or in logs

$$ln(p) + ln(MP_E) = ln(MC_E). \tag{1}$$

Below we derive the input demand functions corresponding to each hours regime.

$\underline{h < h^*}$

The expressions for $ln(MP_E)$ and $ln(MC_E)$ can be obtained as follows:

$$\begin{aligned} ln(MP_E) &= ln(\alpha A) + (\alpha - 1)ln(E) + \beta ln(h) + \gamma ln(K) + gt \\ &= ln(\alpha A) + (\alpha - 1)ln(E) + \beta \left[ln\left(\frac{\beta}{\alpha - \beta}\right) + ln\left(\frac{B}{w}\right)\right] \\ &+ \gamma \left[ln\left(\frac{\gamma}{\alpha - \beta}\right) + ln\left(\frac{B}{r}\right) + ln(E)\right] + gt \\ &= ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha - \beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha - \beta}\right) \\ &- \beta ln\left(\frac{w}{r}\right) + (\beta + \gamma)\, ln\left(\frac{B}{r}\right) + gt + (\alpha + \gamma - 1)ln(E). \end{aligned}$$

$$\begin{aligned} ln(MC_E) &= ln(wh + B) \\ &= ln\left(\frac{\alpha}{\alpha - \beta}\right) + ln\,(B)\,. \end{aligned}$$

The demand for employment (in logs) is obtained by substituting for the above expressions for $ln(MP_E)$ and $ln(MC_E)$ in eq. (1), collecting terms, and solving for $ln(E)$:

$$\begin{aligned} ln(E) = &\left[\frac{1}{1 - (\alpha + \gamma)}\right] \left[ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha - \beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha - \beta}\right) - ln\left(\frac{\alpha}{\alpha - \beta}\right)\right] \\ &- \left[\frac{\beta}{1 - (\alpha + \gamma)}\right] ln\left(\frac{w}{r}\right) + \left[\frac{\beta + \gamma}{1 - (\alpha + \gamma)}\right] ln\left(\frac{B}{r}\right) - \left[\frac{1}{1 - (\alpha + \gamma)}\right] ln\left(\frac{B}{p}\right) \\ &+ \left[\frac{g}{1 - (\alpha + \gamma)}\right] t. \end{aligned} \tag{2}$$

The input demand function for K is obtained by taking the log of eq. (**??**), substituting eq. (2) for $ln(E)$, and collecting terms:

$$\begin{aligned} ln(K) = &\left[\frac{1}{1 - (\alpha + \gamma)}\right] \left[ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha - \beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha - \beta}\right) - ln\left(\frac{\alpha}{\alpha - \beta}\right)\right] \\ &+ ln\left(\frac{\gamma}{\alpha - \beta}\right) - \left[\frac{\beta}{1 - (\alpha + \gamma)}\right] ln\left(\frac{w}{r}\right) + \left[\frac{1 - (\alpha - \beta)}{1 - (\alpha + \gamma)}\right] ln\left(\frac{B}{r}\right) \\ &- \left[\frac{1}{1 - (\alpha + \gamma)}\right] ln\left(\frac{B}{p}\right) + \left[\frac{g}{1 - (\alpha + \gamma)}\right] t. \end{aligned} \tag{3}$$

The partial effects of the variables w, B, r, p, and t on the input demands for employment and capital are derived below.

$$\frac{\partial ln(E)}{\partial ln(w)} = \frac{-\beta}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(E)}{\partial ln(B)} = \frac{\beta+\gamma-1}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(E)}{\partial ln(r)} = \frac{-\gamma}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(E)}{\partial ln(p)} = \frac{1}{1-(\alpha+\gamma)} > 0$$
$$\frac{\partial ln(E)}{\partial t} = \frac{g}{1-(\alpha+\gamma)} > 0.$$

$$\frac{\partial ln(K)}{\partial ln(w)} = \frac{-\beta}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(K)}{\partial ln(B)} = \frac{-(\alpha-\beta)}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(K)}{\partial ln(r)} = \frac{\alpha-1}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(K)}{\partial ln(p)} = \frac{1}{1-(\alpha+\gamma)} > 0$$
$$\frac{\partial ln(K)}{\partial t} = \frac{g}{1-(\alpha+\gamma)} > 0.$$

The partial effects of w, B, r, p, and t on aggregate/total hours demanded (H) are given below:

$$\frac{\partial ln(H)}{\partial ln(w)} = \frac{\partial ln(h)}{\partial ln(w)} + \frac{\partial ln(E)}{\partial ln(w)}$$
$$= -1 - \frac{\beta}{1-(\alpha+\gamma)}$$
$$= \frac{\alpha+\gamma-\beta-1}{1-(\alpha+\gamma)} < 0$$

$$\frac{\partial ln(H)}{\partial ln(B)} = \frac{\partial ln(h)}{\partial ln(B)} + \frac{\partial ln(E)}{\partial ln(B)}$$
$$= 1 + \frac{\beta+\gamma-1}{1-(\alpha+\gamma)}$$
$$= \frac{-(\alpha-\beta)}{1-(\alpha+\gamma)} < 0$$

$$\frac{\partial ln(H)}{\partial ln(r)} = \frac{\partial ln(h)}{\partial ln(r)} + \frac{\partial ln(E)}{\partial ln(r)}$$
$$= 0 - \frac{\gamma}{1-(\alpha+\gamma)}$$
$$= -\frac{\gamma}{1-(\alpha+\gamma)} < 0$$

$$\frac{\partial ln(H)}{\partial ln(p)} = \frac{\partial ln(h)}{\partial ln(p)} + \frac{\partial ln(E)}{\partial ln(p)}$$
$$= 0 + \frac{1}{1-(\alpha+\gamma)}$$
$$= \frac{1}{1-(\alpha+\gamma)} > 0$$

$$\frac{\partial ln(H)}{\partial t} = \frac{\partial ln(h)}{\partial t} + \frac{\partial ln(E)}{\partial t}$$
$$= 0 + \frac{g}{1-(\alpha+\gamma)}$$
$$= \frac{g}{1-(\alpha+\gamma)} > 0.$$

The signs of the partial effects of the variables w, B, r, p, and t on the input demands are summarized in the table below.

$h < h^*$				
	h	E	K	H
w	-	-	-	-
B	+	-	-	-
r	0	-	-	-
p	0	+	+	+
t	0	+	+	+

$\underline{h > h^*}$

After a little algebra it will be helpful to use the following equivalent expressions for $ln(MC_E)$ and $ln(MP_E)$ in the overtime regime:

$$ln(MC_E) = ln(w) + ln\left(\frac{\alpha}{\alpha - \beta}\right) + ln\left[(1-\lambda)\,h^* + \frac{B}{w}\right]$$

$$\begin{aligned} ln(MP_E) = ln(\alpha A) + (\beta)\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) + \gamma ln\left(\frac{w}{r}\right) \\ + (\beta+\gamma)\,ln\left[(1-\lambda)\,h^* + \frac{B}{w}\right] + gt - [1-(\alpha+\gamma)]\,ln(E). \end{aligned}$$

The demand for employment (in logs) is obtained by substituting the appropriate expressions for $ln(MP_E)$ and $ln(MC_E)$ for the overtime regime in equation (1), collecting terms, and solving for $ln(E)$:

$$\begin{aligned} ln(E) = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{ln(\alpha A) + \beta\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right\} \\ + \left[\frac{\gamma}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{r}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{p}\right) \\ + \left[\frac{\beta+\gamma-1}{1-(\alpha+\gamma)}\right] ln\left[(1-\lambda)\,h^* + \frac{B}{w}\right] + \left[\frac{g}{1-(\alpha+\gamma)}\right] t. \end{aligned} \quad (4)$$

The input demand function for $ln(K)$ is obtained substituting (4) for $ln(E)$ in (**??**) and collecting terms:

$$\begin{aligned}
ln(K) = & \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{ln(\alpha A) + (\beta)\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right\} \\
& + ln\left(\frac{\gamma}{\alpha-\beta}\right) + \left[\frac{1-\alpha}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{r}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{p}\right) \\
& + \left[\frac{\beta-\alpha}{1-(\alpha+\gamma)}\right] ln\left[(1-\lambda)\, h^{*} + \frac{B}{w}\right] + \left[\frac{g}{1-(\alpha+\gamma)}\right] t.
\end{aligned}$$

The partial effects of the variables $w, B, r, p, t, \lambda,$ and h^* on the long-run profit maximizing input demand functions in an overtime regime are derived below.

$$\frac{\partial ln(E)}{\partial ln(w)} = w\frac{\partial ln(E)}{\partial w}$$
$$= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\gamma - 1 + \frac{1-(\beta+\gamma)}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) \lesseqgtr 0$$
$$\frac{\partial ln(E)}{\partial ln(B)} = B\frac{\partial ln(E)}{\partial B}$$
$$= \left[\frac{\beta+\gamma-1}{1-(\alpha+\gamma)}\right]\left[\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) < 0$$
$$\frac{\partial ln(E)}{\partial ln(r)} = \left[\frac{-\gamma}{1-(\alpha+\gamma)}\right] < 0$$
$$\frac{\partial ln(E)}{\partial ln(p)} = \left[\frac{1}{1-(\alpha+\gamma)}\right] > 0$$
$$\frac{\partial ln(E)}{\partial t} = \left[\frac{g}{1-(\alpha+\gamma)}\right] > 0$$
$$\frac{\partial ln(E)}{\partial ln(\lambda)} = \lambda\frac{\partial ln(E)}{\partial \lambda}$$
$$= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{\frac{[1-(\beta+\gamma)]\,(\lambda h^*)}{(1-\lambda)\,h^* + \dfrac{B}{w}} - \beta\right\} \lesseqgtr 0$$
$$\frac{\partial ln(E)}{\partial ln(h^*)} = h^*\frac{\partial ln(E)}{\partial h^*}$$
$$= \left[\frac{\beta+\gamma-1}{1-(\alpha+\gamma)}\right]\left[\frac{(1-\lambda)\,h^*}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right] > 0.$$

$$
\begin{aligned}
\frac{\partial ln(K)}{\partial ln(w)} &= w\frac{\partial ln(K)}{\partial w} \\
&= \left[\frac{-1}{1-(\alpha+\gamma)}\right]\left\{\alpha+\left[\frac{\beta+\gamma}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]\left(\frac{B}{w}\right)\right\}<0 \\
\frac{\partial ln(K)}{\partial ln(B)} &= B\frac{\partial ln(K)}{\partial B} \\
&= \left[\frac{\beta-\alpha}{1-(\alpha+\gamma)}\right]\left[\frac{1}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]\left(\frac{B}{w}\right)<0 \\
\frac{\partial ln(K)}{\partial ln(r)} &= \left[\frac{-(1-\alpha)}{1-(\alpha+\gamma)}\right]<0 \\
\frac{\partial ln(K)}{\partial ln(p)} &= \left[\frac{1}{1-(\alpha+\gamma)}\right]>0 \\
\frac{\partial ln(K)}{\partial t} &= \left[\frac{g}{1-(\alpha+\gamma)}\right]>0 \\
\frac{\partial ln(K)}{\partial ln(\lambda)} &= \lambda\frac{\partial ln(K)}{\partial \lambda} \\
&= \left[\frac{-1}{1-(\alpha+\gamma)}\right]\left[\beta+\frac{(\beta-\alpha)\,(\lambda h^{*})}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]<0 \\
\frac{\partial ln(K)}{\partial ln(h^{*})} &= h^{*}\frac{\partial ln(K)}{\partial h^{*}} \\
&= \left[\frac{\beta-\alpha}{1-(\alpha+\gamma)}\right]\left[\frac{(1-\lambda)\,h^{*}}{(1-\lambda)\,h^{*}+\dfrac{B}{w}}\right]<0.
\end{aligned}
$$

The partial effects of the variables w, B, r, p, t, λ, and h^* on aggregate/total hours demanded (H) are given below:

$$\begin{aligned}
\frac{\partial ln(H)}{\partial ln(w)} &= \frac{\partial ln(h)}{\partial ln(w)} + \frac{\partial ln(E)}{\partial ln(w)} \\
&= \left(\frac{-1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right) + \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\gamma - 1 + \frac{1-(\beta+\gamma)}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) \\
&= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\gamma - 1 + \frac{\alpha-\beta}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) \lesseqgtr 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial ln(H)}{\partial ln(B)} &= \frac{\partial ln(h)}{\partial ln(B)} + \frac{\partial ln(E)}{\partial ln(B)} \\
&= \left(\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right)\left(\frac{B}{w}\right) + \left[\frac{\beta+\alpha-1}{1-(\alpha+\gamma)}\right]\left[\frac{1}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) \\
&= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\frac{\beta-\alpha}{(1-\lambda)\,h^* + \dfrac{B}{w}}\right]\left(\frac{B}{w}\right) < 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial ln(H)}{\partial ln(r)} &= \frac{\partial ln(h)}{\partial ln(r)} + \frac{\partial ln(E)}{\partial ln(r)} \\
&= 0 - \frac{\gamma}{1-(\alpha+\gamma)} \\
&= -\frac{\gamma}{1-(\alpha+\gamma)} < 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial ln(H)}{\partial ln(p)} &= \frac{\partial ln(h)}{\partial ln(p)} + \frac{\partial ln(E)}{\partial ln(p)} \\
&= 0 + \frac{1}{1-(\alpha+\gamma)} \\
&= \frac{1}{1-(\alpha+\gamma)} > 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial t} &= \frac{\partial ln(h)}{\partial t} + \frac{\partial ln(E)}{\partial t} \\
&= 0 + \frac{g}{1-(\alpha+\gamma)} \\
&= \frac{g}{1-(\alpha+\gamma)} > 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial ln(\lambda)} &= \frac{\partial ln(h)}{\partial ln(\lambda)} + \frac{\partial ln(E)}{\partial ln\,(\lambda)} \\
&= -\left[1+\left(\frac{\lambda h^*}{(1-\lambda)\,h^* + \frac{B}{w}}\right)\right] + \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{\frac{[1-(\beta+\gamma)]\,(\lambda h^*)}{(1-\lambda)\,h^*+\frac{B}{w}} - \beta\right\} \\
&= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{(\alpha-\beta)+(\gamma-1)+\frac{(\alpha-\beta)\,(\lambda h^*)}{(1-\lambda)\,h^*+\frac{B}{w}}\right\} \lesseqgtr 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial ln(H)}{\partial ln(h^*)} &= \frac{\partial ln(h)}{\partial ln(h^*)} + \frac{\partial ln(E)}{\partial ln\,(h^*)} \\
&= \left[\frac{(1-\lambda)\,h^*}{(1-\lambda)\,h^*+\frac{B}{w}}\right] + \left[\frac{\beta+\gamma-1}{1-(\alpha+\gamma)}\right]\left[\frac{(1-\lambda)\,h^*}{(1-\lambda)\,h^*+\frac{B}{w}}\right] \\
&= \left[\frac{1-(\alpha-\beta)}{1-(\alpha+\gamma)}\right]\left[\frac{(1-\lambda)\,h^*}{(1-\lambda)\,h^*+\frac{B}{w}}\right] < 0
\end{aligned}
$$

The signs of the partial effects of the variables on input demands are summarized in the table below.

$h > h^*$				
	h	E	K	H
w	-	?	-	?
B	+	-	-	-
r	0	-	-	-
p	0	+	+	+
t	0	+	+	+
λ	-	?	-	-
h^*	-	+	-	-

The last case to be considered is that of the standard workweek.

$\underline{h = h^*}$

After a little algebra it will be helpful to use the following equivalent expressions for $ln(MC_E)$ and $ln(MP_E)$ in the standard hours regime:

$$ln(MP_E) = ln\,(\alpha A) + (\alpha + \gamma - 1)\,ln(E) + \beta ln(h^*) + \gamma ln\left(\frac{w}{r}\right) + \gamma ln\left(h^* + \frac{B}{w}\right) + gt$$

$$ln(MC_E) = ln(wh^* + B) = ln(w) + ln\left(h^* + \frac{B}{w}\right).$$

The demand for employment (in logs) is obtained by substituting the appropriate expressions for $ln(MP_E)$ and $ln(MC_E)$ for the standard hours regime in equation (1), collecting terms, and solving for $ln(E)$:

$$
\begin{aligned}
ln(E) = & \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \gamma ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right] + \left[\frac{\gamma}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{r}\right) \\
& - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{p}\right) + \left[\frac{\gamma-1}{1-(\alpha+\gamma)}\right] ln\left(h^* + \frac{B}{w}\right) \\
& + \left[\frac{g}{1-(\alpha+\gamma)}\right] t. \qquad (5)
\end{aligned}
$$

The input demand function for $ln(K)$ is obtained substituting (4) for $ln(E)$ in (**??**) and collecting terms:

$$
\begin{aligned}
ln(K) = & \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + (1-\alpha)\, ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right] + \left[\frac{1-\alpha}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{r}\right) \\
& - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{p}\right) - \left[\frac{\alpha}{1-(\alpha+\gamma)}\right] ln\left(h^* + \frac{B}{w}\right) + \left[\frac{g}{1-(\alpha+\gamma)}\right] t. \qquad (6)
\end{aligned}
$$

The partial effects of the variables w, B, r, p, t, and h^* on the long-run profit maximizing input demands in the standard hours regime are derived below.

$$\begin{aligned}
\frac{\partial ln(E)}{\partial ln(w)} &= w\frac{\partial ln(E)}{\partial w} \\
&= \left[\frac{\gamma - 1}{1-(\alpha+\gamma)}\right]\left[1 - \left(\frac{1}{h^* + \frac{B}{w}}\right)\left(\frac{B}{w}\right)\right] < 0 \\
\frac{\partial ln(E)}{\partial ln(B)} &= B\frac{\partial ln(E)}{\partial B} \\
&= \left[\frac{\gamma - 1}{1-(\alpha+\gamma)}\right]\left(\frac{1}{h^* + \frac{B}{w}}\right)\left(\frac{B}{w}\right) < 0 \\
\frac{\partial ln(E)}{\partial ln(r)} &= \frac{\alpha - 1}{1-(\alpha+\gamma)} < 0 \\
\frac{\partial ln(E)}{\partial ln(p)} &= \left[\frac{1}{1-(\alpha+\gamma)}\right] > 0 \\
\frac{\partial ln(E)}{\partial t} &= \left[\frac{g}{1-(\alpha+\gamma)}\right] > 0 \\
\frac{\partial ln(E)}{\partial ln(h^*)} &= h^*\frac{\partial ln(E)}{\partial h^*} \\
&= \left[\frac{\gamma - 1}{1-(\alpha+\gamma)}\right]\left(\frac{h^*}{h^* + \frac{B}{w}}\right) < 0.
\end{aligned}$$

$$\frac{\partial ln(K)}{\partial ln(w)} = w\frac{\partial ln(K)}{\partial w}$$
$$= \left[\frac{\alpha}{1-(\alpha+\gamma)}\right]\left[-1+\left(\frac{\alpha}{h^*+\dfrac{B}{w}}\right)\left(\frac{B}{w}\right)\right] < 0$$
$$\frac{\partial ln(K)}{\partial ln(B)} = B\frac{\partial ln(K)}{\partial B}$$
$$= \left[\frac{-\alpha}{1-(\alpha+\gamma)}\right]\left(\frac{1}{h^*+\dfrac{B}{w}}\right)\left(\frac{B}{w}\right) < 0$$
$$\frac{\partial ln(K)}{\partial ln(r)} = \frac{\alpha-1}{1-(\alpha+\gamma)} < 0$$
$$\frac{\partial ln(K)}{\partial ln(p)} = \left[\frac{1}{1-(\alpha+\gamma)}\right] > 0$$
$$\frac{\partial ln(K)}{\partial t} = \left[\frac{g}{1-(\alpha+\gamma)}\right] > 0$$
$$\frac{\partial ln(K)}{\partial ln(h^*)} = h^*\frac{\partial ln(K)}{\partial h^*}$$
$$= \left[\frac{-\alpha}{1-(\alpha+\gamma)}\right]\left(\frac{1}{h^*+\dfrac{B}{w}}\right) < 0.$$

The partial effects of the variables $w, B, r, p, t,$ and h^* on aggregate/total hours demanded ($H^* = h^*E$)(assuming that the actual work week changes to match the changed new standard workweek) are given below:

$$\frac{\partial ln(H^*)}{\partial ln(w)} = \frac{\partial ln(E)}{\partial ln(w)}$$
$$= \left[\frac{\gamma-1}{1-(\alpha+\gamma)}\right]\left[1-\left(\frac{1}{h^*+\dfrac{B}{w}}\right)\left(\frac{B}{w}\right)\right] < 0$$

$$\frac{\partial ln(H^*)}{\partial ln(B)} = \frac{\partial ln(E)}{\partial ln(B)}$$
$$= \left[\frac{\beta+\alpha-1}{1-(\alpha+\gamma)}\right]\left[\frac{1}{(1-\lambda)\,h^*+\dfrac{B}{w}}\right]\left(\frac{B}{w}\right) < 0$$

$$\frac{\partial ln(H^*)}{\partial ln(r)} = \frac{\partial ln(E)}{\partial ln(r)}$$
$$= \frac{-\gamma}{1-(\alpha+\gamma)} < 0$$

$$\frac{\partial ln(H^*)}{\partial ln(p)} = \frac{\partial ln(E)}{\partial ln(p)}$$
$$= \frac{1}{1-(\alpha+\gamma)} > 0$$

$$\frac{\partial ln(H^*)}{\partial t} = \frac{\partial ln(E)}{\partial t}$$
$$= \frac{g}{1-(\alpha+\gamma)} > 0$$

$$\frac{\partial ln(H^*)}{\partial ln(h^*)} = \frac{\partial ln(h^*)}{\partial ln(h^*)} + \frac{\partial ln(E)}{\partial ln(h^*)}$$
$$= 1 + \left[\frac{\gamma-1}{1-(\alpha+\gamma)}\right]\left(\frac{h^*}{h^*+\frac{B}{w}}\right) < 0.$$

The signs of the partial effects of the variables on input demands are summarized in the table below.

$h = h^*$			
	E	K	H
w	-	-	-
B	-	-	-
r	-	-	-
p	+	+	+
t	+	+	+
h^*	-	-	-

Again it should be clear that changes in the ratio B/w can lead to a regime switch. Which regime is profit maximizing is determined according to

$$max\left[\pi\left(w,B,r,p,h \mid h<h^*\right), \pi\left(w,B,r,p,h \mid h>h^*\right), \pi\left(w,B,r,p,h \mid h=h^*\right)\right],$$

which corresponds to

$$\frac{B}{w} < \left(\frac{\alpha}{\beta} - 1\right) h^*$$

$$\frac{B}{w} > \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*$$

$$\left(\frac{\alpha}{\beta} - 1\right) h^* \leq \frac{B}{w} \leq \left(\frac{\alpha}{\beta}\lambda - 1\right) h^*.$$

II. COBB-DOUGLAS LONG-RUN PROFIT MAXIMIZATION - EMPIRICAL

We now consider the empirical estimation of the long-run profit maximizing input demand functions. We begin with the empirical specification of the input demand functions when there is no overtime. As will be shown below, as long as we have data on w, r, B, and p it is possible to estimate all of the parameters of the CD technology and the long-run profit maximizing input demand function parameters from data on employment alone. Although the overtime premium λ is treated as a parameter it could be treated as a variable if it were to change.

$\underline{h < h^*}$

$$ln(h_t) - ln\left(\frac{B_t}{w_t}\right) = a_{01} + \varepsilon_{h1t}$$

$$ln\left(E_t\right) = b_{01} + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) + b_{31}ln\left(\frac{B_t}{p_t}\right) + b_{41}t + \varepsilon_{E1t}$$

$$ln\left(K_t\right) = c_{01} + c_{11}ln\left(\frac{w_t}{r_t}\right) + c_{21}ln\left(\frac{B_t}{r_t}\right) + c_{31}ln\left(\frac{B_t}{p_t}\right) + c_{41}t + \varepsilon_{K1t}.$$

From the theoretical model we know that there are within and cross-equation restrictions on the parameters:

$$b_{01} = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right]$$

$$b_{11} = \frac{-\beta}{1-(\alpha+\gamma)} < 0$$

$$b_{21} = \frac{\beta+\gamma}{1-(\alpha+\gamma)} > 0$$

$$b_{31} = \frac{-1}{1-(\alpha+\gamma)} < 0$$

$$b_{41} = \frac{g}{1-(\alpha+\gamma)} > 0$$

$$c_{01} = b_{01} + ln\left(\frac{\gamma}{\alpha - \beta}\right) = b_{01} + ln\left[\frac{-(b_{11} + b_{21})}{1 + b_{21} + b_{31}}\right]$$

$$c_{11} = b_{11} < 0$$

$$c_{21} = \ 1 + b_{21} > 0$$

$$c_{31} = b_{31} < 0$$

$$c_{41} = b_{41} > 0$$

$$a_{01} = ln\left(\frac{\beta}{\alpha - \beta}\right) = ln\left(\frac{b_{11}}{1 + b_{21} + b_{31}}\right)$$

The resulting estimating equations incorporating these restrictions are as follows:

$$ln(h_t) - ln\left(\frac{B_t}{w_t}\right) = ln\left(\frac{b_{11}}{1 + b_{21} + b_{31}}\right) + \varepsilon_{h1t} \tag{7}$$

$$ln\left(E_t\right) = b_{01} + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) + b_{31}ln\left(\frac{B_t}{p_t}\right) + b_{41}t + \varepsilon_{E1t} \tag{8}$$

$$\begin{aligned} ln\left(K_t\right) - ln\left(\frac{B_t}{r_t}\right) &= b_{01} + +ln\left[\frac{-(b_{11} + b_{21})}{1 + b_{21} + b_{31}}\right] + b_{11}ln\left(\frac{w_t}{r_t}\right) + b_{21}ln\left(\frac{B_t}{r_t}\right) \\ &+ b_{31}\left(\frac{B_t}{p_t}\right) + b_{41}t + \varepsilon_{K1t}. \end{aligned} \tag{9}$$

The model could be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions.

The parameters of the underlying CD technology are identified and can be recovered from the estimated conditional input demand function parameters:

$$\tilde{A} = exp\ \left\{ -ln\left(\frac{1+\hat{b}_{11}+\hat{b}_{21}+\hat{b}_{31}}{\hat{b}_{31}}\right) - \frac{\hat{b}_{01}}{\hat{b}_{31}} - \frac{\hat{b}_{11}}{\hat{b}_{31}} ln\left(\frac{1+\hat{b}_{11}}{1+\hat{b}_{21}+\hat{b}_{31}}\right) \right.$$
$$\left. +\left(\frac{\hat{b}_{11}+\hat{b}_{21}}{\hat{b}_{31}}\right) ln\left[\frac{-\left(\hat{b}_{11}+\hat{b}_{21}\right)}{1+\hat{b}_{21}+\hat{b}_{31}}\right] + ln\left(\frac{1+\hat{b}_{11}+\hat{b}_{21}+\hat{b}_{31}}{1+\hat{b}_{21}+\hat{b}_{31}}\right) \right\}$$
$$\tilde{\alpha} = \frac{1+\hat{b}_{11}+\hat{b}_{21}+\hat{b}_{31}}{\hat{b}_{31}}$$
$$\tilde{\beta} = \frac{\hat{b}_{11}}{\hat{b}_{31}}$$
$$\tilde{\gamma} = \frac{-\left(\hat{b}_{11}+\hat{b}_{21}\right)}{\hat{b}_{31}}$$
$$\tilde{g} = \frac{-\hat{b}_{41}}{\hat{b}_{31}}.$$

All of the parameters of the conditional input demand functions as well as of the CD production function can be estimated from the demand for employment equation alone (8). While not fully efficient, this strategy is feasible if data on h_t and K_t were unavailable.

An alternative estimation strategy is to directly estimate the CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$ln(h_t) - ln\left(\frac{B_t}{w_t}\right) = ln\left(\frac{\beta}{\alpha-\beta}\right) + \varepsilon_{h1t}$$

$$\begin{aligned}
ln\,(E_t) = & \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right] \\
& - \left[\frac{\beta}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{r_t}\right) + \left[\frac{\beta+\gamma}{1-(\alpha+\gamma)}\right] ln\left(\frac{B_t}{r_t}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{B_t}{p_t}\right) \\
& + \left[\frac{g}{1-(\alpha+\gamma)}\right] t + \varepsilon_{E1t} \\
ln(K_t) = & \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \beta ln\left(\frac{\beta}{\alpha-\beta}\right) + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right] \\
& + ln\left(\frac{\gamma}{\alpha-\beta}\right) - \left[\frac{\beta}{1-(\alpha+\gamma)}\right] ln\left(\frac{w}{r}\right) \\
& + \left[\frac{1-(\alpha-\beta)}{1-(\alpha+\gamma)}\right] ln\left(\frac{B}{r}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{B}{p}\right) \\
& + \left[\frac{g}{1-(\alpha+\gamma)}\right] t + \varepsilon_{K1t}.
\end{aligned}$$

$$\begin{aligned}
\tilde{b}_{01} &= \left[\frac{1}{1-(\hat{\alpha}+\hat{\gamma})}\right]\left[ln(\hat{\alpha}\hat{A}) + \hat{\beta} ln\left(\frac{\hat{\beta}}{\hat{\alpha}-\hat{\beta}}\right) + \hat{\gamma} ln\left(\frac{\hat{\gamma}}{\hat{\alpha}-\hat{\beta}}\right) - ln\left(\frac{\hat{\alpha}}{\hat{\alpha}-\hat{\beta}}\right)\right] \\
\tilde{b}_{11} &= \frac{-\hat{\beta}}{1-(\hat{\alpha}+\hat{\gamma})} \\
\tilde{b}_{21} &= \frac{\hat{\beta}+\hat{\gamma}}{1-(\hat{\alpha}+\hat{\gamma})} \\
\tilde{b}_{31} &= \left[\frac{-1}{1-(\hat{\alpha}+\hat{\gamma})}\right] \\
\tilde{b}_{41} &= \frac{\hat{g}}{1-(\hat{\alpha}+\hat{\gamma})} \\
\tilde{c}_{01} &= \tilde{b}_{01} + ln\left(\frac{\hat{\gamma}}{\hat{\alpha}-\hat{\beta}}\right) \\
\tilde{c}_{11} &= \tilde{b}_{11} \\
\tilde{c}_{21} &= 1 + \tilde{b}_{21} \\
\tilde{c}_{31} &= \tilde{b}_{31} \\
\tilde{c}_{41} &= \tilde{b}_{41} \\
\tilde{a}_{01} &= ln\left(\frac{\tilde{b}_{11}}{1+\tilde{b}_{21}+\tilde{b}_{31}}\right).
\end{aligned}$$

Next, we consider estimation of the long-run profit maximizing input labor demands in the overtime hours regime.

$\underline{h > h^*}$

$$ln(h_t) - ln\left[(1-\lambda)\,h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = a_{02} + \ \varepsilon_{h2t}$$

$$\begin{aligned} ln(E_t) &= b_{02} + b_{12} ln\left(\frac{w_t}{r_t}\right) + b_{22}\ ln\left(\frac{w_t}{p_t}\right) \\ &+ b_{32} ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] + b_{42} t + \varepsilon_{E2t} \\ ln(K_t) &= c_{02} + c_{12} ln\left(\frac{w_t}{r_t}\right) + c_{22} ln\left(\frac{w_t}{p_t}\right) \\ &+ c_{32} ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] + c_{42} t + \varepsilon_{K2t}. \end{aligned}$$

From the theoretical model we know that there are within and cross-equation restrictions on the parameters:

$$\begin{aligned} b_{02} &= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{ln(\alpha A) + \beta\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right\} \\ b_{12} &= \frac{\gamma}{1-(\alpha+\gamma)} > 0 \\ b_{22} &= \frac{-1}{1-(\alpha+\gamma)} < 0 \\ b_{32} &= \frac{\beta+\gamma-1}{1-(\alpha+\gamma)} < 0 \\ b_{42} &= \frac{g}{1-(\alpha+\gamma)} > 0 \end{aligned}$$

$$c_{02} = b_{02} + ln\left(\frac{\gamma}{\alpha-\beta}\right) = b_{02} + ln\left(\frac{-b_{12}}{1+b_{32}}\right)$$
$$c_{12} = \frac{1-\alpha}{1-(\alpha+\gamma)} = 1 + b_{12} > 0$$
$$c_{22} = \frac{-1}{1-(\alpha+\gamma)} = b_{22} < 0$$
$$c_{32} = \frac{\beta-\alpha}{1-(\alpha+\gamma)} = 1 + b_{32} < 0$$
$$c_{42} = \frac{g}{1-(\alpha+\gamma)} = b_{42} > 0$$
$$a_{02} = ln\left(\frac{\beta}{\alpha-\beta}\right) = ln\left(\frac{b_{12}+b_{22}-b_{32}}{1+b_{32}}\right)$$

The resulting estimating equations incorporating these restrictions are as follows:

$$ln(h_t) - ln\left[(1-\lambda)\,h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = ln\left(\frac{b_{12}+b_{22}-b_{32}}{1+b_{32}}\right) + \ \varepsilon_{h2t} \qquad (10)$$

$$ln(E_t) = b_{02} + b_{12}ln\left(\frac{w_t}{r_t}\right) + b_{22}\ ln\left(\frac{w_t}{p_t}\right) + b_{32}ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] + b_{42}t + \varepsilon_{E2t} \quad (11)$$

$$ln(K_t) - ln\left(\frac{w_t}{r_t}\right) - ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] = b_{02} + ln\left(\frac{-b_{12}}{1+b_{32}}\right) + b_{12}ln\left(\frac{w_t}{r_t}\right) + b_{22}ln\left(\frac{w_t}{p_t}\right) + b_{32}ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] + b_{42}t + \varepsilon_{K2t}. \qquad (12)$$

The model can be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions. Also, the parameters of the underlying CD technology are identified and can be recovered from the estimated long-run profit maximizing input demand function parameters:

$$\tilde{A} = exp\ \left\{ \left(\frac{\hat{b}_{32} - \hat{b}_{12} - \hat{b}_{22}}{\hat{b}_{22}}\right) \left[ln\left(\frac{\hat{b}_{32} - \hat{b}_{12} - \hat{b}_{22}}{1 + \hat{b}_{32}}\right) - ln(\lambda) \right] + \left(\frac{\hat{b}_{12}}{\hat{b}_{22}}\right) ln\left(\frac{-\hat{b}_{12}}{1 + \hat{b}_{32}}\right) + ln\left(\frac{\hat{b}_{22}}{1 + \hat{b}_{32}}\right) - \left(\frac{\hat{b}_{02}}{\hat{b}_{22}}\right) \right\}$$

$$\tilde{\alpha} = \frac{1 + \hat{b}_{12} + \hat{b}_{22}}{\hat{b}_{22}}$$

$$\tilde{\beta} = \frac{\hat{b}_{12} + \hat{b}_{22} - \hat{b}_{32}}{\hat{b}_{22}}$$

$$\tilde{\gamma} = \frac{-\hat{b}_{12}}{\hat{b}_{22}}$$

$$\tilde{g} = \frac{-\hat{b}_{42}}{\hat{b}_{22}}.$$

All of the parameters of the conditional input demand functions as well as of the CD production function could be estimated from the demand for employment equation alone (11). While not fully efficient, this strategy would work if data on h_t and K_t were unavailable.

An alternative estimation strategy is to directly estimate the CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$ln(h_t) - ln\left[(1-\lambda)\, h^* + \left(\frac{B_t}{w_t}\right)\right] + ln(\lambda) = ln\left(\frac{\beta}{\alpha - \beta}\right) + \varepsilon_{h2t}$$

$$\begin{aligned} ln(E_t) = & \left[\frac{1}{1-(\alpha+\gamma)}\right] \left\{ ln(\alpha A) + \beta \left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda) \right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right) \right\} \\ & + \left[\frac{\gamma}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{r_t}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{p_t}\right) \\ & + \left[\frac{\beta+\gamma-1}{1-(\alpha+\gamma)}\right] ln\left[(1-\lambda)\, h^* + \frac{B_t}{w_t}\right] + \left[\frac{g}{1-(\alpha+\gamma)}\right] t + \varepsilon_{E2t}. \end{aligned}$$

$$
\begin{aligned}
ln(K_t) &= \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{ln(\alpha A) + (\beta)\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right\} \\
&\quad + ln\left(\frac{\gamma}{\alpha-\beta}\right) + \left[\frac{1-\alpha}{1-(\alpha+\gamma)}\right]ln\left(\frac{w_t}{r_t}\right) - \left[\frac{1}{1-(\alpha+\gamma)}\right]ln\left(\frac{w_t}{p_t}\right) \\
&\quad + \left[\frac{\beta-\alpha}{1-(\alpha+\gamma)}\right]ln\left[(1-\lambda)\,h^* + \frac{B_t}{w_t}\right] + \left[\frac{g}{1-(\alpha+\gamma)}\right]t + \varepsilon_{K2t}.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}_{02} &= \left[\frac{1}{1-(\hat{\alpha}+\hat{\gamma})}\right]\left\{ln(\hat{\alpha}\hat{A}) + \hat{\beta}\left[ln\left(\frac{\hat{\beta}}{\hat{\alpha}-\hat{\beta}}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\hat{\gamma}}{\hat{\alpha}-\hat{\beta}}\right) - ln\left(\frac{\hat{\alpha}}{\hat{\alpha}-\hat{\beta}}\right)\right\} \\
\tilde{b}_{12} &= \frac{\hat{\gamma}}{1-(\hat{\alpha}+\hat{\gamma})} > 0 \\
\tilde{b}_{22} &= \frac{-1}{1-(\hat{\alpha}+\hat{\gamma})} < 0 \\
\tilde{b}_{32} &= \frac{\hat{\beta}+\hat{\gamma}-1}{1-(\hat{\alpha}+\hat{\gamma})} < 0 \\
\tilde{b}_{42} &= \frac{\hat{g}}{1-(\hat{\alpha}+\hat{\gamma})} > 0 \\
\tilde{c}_{02} &= \tilde{b}_{02} + ln\left(\frac{\hat{\gamma}}{\hat{\alpha}-\hat{\beta}}\right) \\
\tilde{c}_{12} &= 1 + \tilde{b}_{12} \\
\tilde{c}_{22} &= \tilde{b}_{22} \\
\tilde{c}_{32} &= 1 + \tilde{b}_{32} \\
\tilde{c}_{42} &= \tilde{b}_{42} \\
\tilde{a}_{02} &= ln\left(\frac{\tilde{b}_{12} + \tilde{b}_{22} - \tilde{b}_{32}}{1 + \tilde{b}_{32}}\right).
\end{aligned}
$$

We now consider estimation of the long-run profit maximizing input labor demands in the standard hours regime.

<u>$h = h^*$</u>

$$ln(E_t) = b_{03} + b_{13} ln\left(\frac{w_t}{r_t}\right) + b_{23} ln\left(\frac{w_t}{p_t}\right) + b_{33} ln\left(h^* + \frac{B_t}{w_t}\right) + b_{43}t + \varepsilon_{E3t}.$$

$$ln(K) = c_{03} + c_{13} ln\left(\frac{w_t}{r_t}\right) + c_{23} ln\left(\frac{w_t}{p_t}\right) + c_{33} ln\left(h^* + \frac{B}{w}\right) + c_{43}t + \varepsilon_{K3t}.$$

The theoretical model implies the following within and cross-equation restrictions on the input demand function parameters:

$$b_{03} = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \gamma ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right] = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\phi + \gamma ln\left(\frac{\gamma}{\alpha}\right)\right],$$

where $\phi = ln(\alpha A) + \beta ln(h^*)$

$$b_{13} = \frac{\gamma}{1-(\alpha+\gamma)} > 0$$

$$b_{23} = \frac{-1}{1-(\alpha+\gamma)} < 0$$

$$b_{33} = \frac{\gamma - 1}{1-(\alpha+\gamma)} < 0$$

$$b_{43} = \frac{g}{1-(\alpha+\gamma)} > 0$$

$$c_{03} = b_{03} + ln\left(\frac{\gamma}{\alpha}\right) = b_{03} + ln\left(\frac{-b_{13}}{1+b_{13}+b_{23}}\right)$$

$$c_{13} = \frac{1-\alpha}{1-(\alpha+\gamma)} = 1 + b_{13} > 0$$

$$c_{23} = \frac{-1}{1-(\alpha+\gamma)} = b_{23} < 0$$

$$c_{33} = \frac{-\alpha}{1-(\alpha+\gamma)} = -\left(1+b_{13}+b_{23}\right) < 0$$

$$c_{43} = \frac{g}{1-(\alpha+\gamma)} = b_{43} > 0$$

The resulting estimating equations incorporating the above restrictions are as follows:

$$ln(E_t) = b_{03} + b_{13}ln\left(\frac{w_t}{r_t}\right) + b_{23}ln\left(\frac{w_t}{p_t}\right) + b_{33}ln\left(h^* + \frac{B_t}{w_t}\right) + b_{43}t + \varepsilon_{E3t}. \qquad (13)$$

$$\begin{aligned} ln(K_t) - ln\left(\frac{w_t}{r_t}\right) &= b_{03} + ln\left(\frac{-b_{13}}{1 + b_{13} + b_{23}}\right) + b_{13}ln\left(\frac{w_t}{r_t}\right) + b_{23}ln\left(\frac{w_t}{p_t}\right) \\ &- (1 + b_{13} + b_{23})\, ln\left(h^* + \frac{B_t}{w_t}\right) + b_{43}t + \varepsilon_{K3t}. \qquad (14) \end{aligned}$$

As in the other hours regimes, the model could be estimated by nonlinear seemingly unrelated regression (NLSUR) with cross-equation restrictions.

The following identified parameters of the underlying CD technology can be recovered from the estimated long-run profit maximizing input demand function parameters:

$$\tilde{\alpha} = \frac{1 + \hat{b}_{13} + \hat{b}_{23}}{\hat{b}_{23}}$$
$$\tilde{\gamma} = \frac{-\hat{b}_{13}}{\hat{b}_{23}}$$
$$\tilde{g} = \frac{-\hat{b}_{43}}{\hat{b}_{23}}$$
$$\tilde{\phi} = \left(\frac{-1}{\hat{b}_{23}}\right)\left[\hat{b}_{03} + \hat{b}_{13}ln\left(\frac{-\hat{b}_{13}}{1 + \hat{b}_{13} + \hat{b}_{23}}\right)\right].$$

Again, the CD production function parameters A and β are not identified from data on just the invariant standard work week. One would need variation in h^* to identify these parameters. Without such variation, all that can be identified is $\phi = ln(A) + \beta ln(h^*)$.

All of the identified parameters of the long-run profit maximizing input demand functions as well as of the CD production function could be estimated from the demand for employment equation alone (13). Although not fully efficient, this strategy would be feasible if data on K_t were unavailable.

As in the case of cost minimization, an alternative estimation strategy is to directly estimate the identified CD production function parameters by NLSUR and recover the conditional input demand function parameters:

$$ln(E_t) = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\phi + \gamma ln\left(\frac{\gamma}{\alpha}\right)\right] + \left[\frac{\gamma}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{r_t}\right)$$
$$- \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{p_t}\right) + \left[\frac{\gamma-1}{1-(\alpha+\gamma)}\right] ln\left(h^* + \frac{B_t}{w_t}\right) + \left[\frac{g}{1-(\alpha+\gamma)}\right] t + \varepsilon_{E3t}.$$

$$ln(K_t) = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[\phi + (1-\alpha)\, ln\left(\frac{\gamma}{\alpha}\right)\right] + \left[\frac{1-\alpha}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{r_t}\right)$$
$$- \left[\frac{1}{1-(\alpha+\gamma)}\right] ln\left(\frac{w_t}{p_t}\right) - \left[\frac{\alpha}{1-(\alpha+\gamma)}\right] ln\left(h^* + \frac{B_t}{w_t}\right) + \left[\frac{g}{1-(\alpha+\gamma)}\right] t + \varepsilon_{K3t}.$$

$$\tilde{b}_{03} = \left[\frac{1}{1-(\hat{\alpha}+\hat{\gamma})}\right]\left[\hat{\phi} + \hat{\gamma} ln\left(\frac{\hat{\gamma}}{\hat{\alpha}}\right)\right]$$
$$\tilde{b}_{13} = \frac{\hat{\gamma}}{1-(\hat{\alpha}+\hat{\gamma})} > 0$$
$$\tilde{b}_{23} = \frac{-1}{1-(\hat{\alpha}+\hat{\gamma})} < 0$$
$$\tilde{b}_{33} = \frac{\hat{\gamma}-1}{1-(\hat{\alpha}+\hat{\gamma})} < 0$$
$$\tilde{b}_{43} = \frac{\hat{g}}{1-(\hat{\alpha}+\hat{\gamma})} > 0$$
$$\tilde{c}_{03} = b_{03} + ln\left(\frac{\gamma}{\alpha}\right) = \tilde{b}_{03} + ln\left(\frac{-\tilde{b}_{13}}{1+\tilde{b}_{13}+\tilde{b}_{23}}\right)$$
$$\tilde{c}_{13} = \frac{1-\hat{\alpha}}{1-(\hat{\alpha}+\hat{\gamma})} = 1 + \tilde{b}_{13} > 0$$
$$\tilde{c}_{23} = \frac{-1}{1-(\hat{\alpha}+\hat{\gamma})} = \tilde{b}_{23} < 0$$
$$\tilde{c}_{33} = \frac{-\hat{\alpha}}{1-(\hat{\alpha}+\hat{\gamma})} = -\left(1 + \tilde{b}_{13} + \tilde{b}_{23}\right) < 0$$
$$\tilde{c}_{43} = \frac{\hat{g}}{1-(\hat{\alpha}+\hat{\gamma})} = \tilde{b}_{43} > 0$$

If the available data spans all three hours regimes and for each observation it is known which regime is in effect, then all of the long-run profit maximizing input demand functions could be jointly estimated with cross-equation restrictions.

Joint Estimation Across Hours Regimes

We note below the cross-equation, conditional input demand function parameter restrictions across hours regimes.

$$b_{02} = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left\{ln(\alpha A) + \beta\left[ln\left(\frac{\beta}{\alpha-\beta}\right) - ln(\lambda)\right] + \gamma ln\left(\frac{\gamma}{\alpha-\beta}\right) - ln\left(\frac{\alpha}{\alpha-\beta}\right)\right\}$$
$$= b_{01} + (1-b_{31})ln\left(\frac{1+b_{11}+b_{21}+b_{31}}{b_{31}}\right) + b_{11}ln\left[\frac{(1+b_{11})(\lambda)}{b_{11}}\right]$$
$$b_{12} = \frac{\gamma}{1-(\alpha+\gamma)} = b_{11} + b_{21}$$
$$b_{22} = \frac{-1}{1-(\alpha+\gamma)} = b_{31}$$
$$b_{32} = \frac{\beta+\gamma-1}{1-(\alpha+\gamma)} = b_{21} + b_{31}$$
$$b_{42} = \frac{g}{1-(\alpha+\gamma)} = b_{41}$$
$$c_{02} = b_{02} + ln\left(\frac{-b_{12}}{1+b_{32}}\right)$$
$$= b_{01} + (1-b_{31})ln\left(\frac{1+b_{11}+b_{21}+b_{31}}{b_{31}}\right) + b_{11}ln\left[\frac{(1+b_{11})(\lambda)}{b_{11}}\right] + ln\left[\frac{-(b_{11}+b_{12})}{1+b_{21}+b_{31}}\right]$$
$$c_{12} = 1 + b_{12} = 1 + b_{11} + b_{21}$$
$$c_{22} = b_{22} = b_{31}$$
$$c_{32} = 1 + b_{32} = 1 + b_{21} + b_{31}$$
$$c_{42} = b_{42} = b_{41}$$
$$a_{02} = ln\left(\frac{b_{12}+b_{22}-b_{32}}{1+b_{32}}\right) = ln\left(\frac{b_{11}}{1+b_{21}+b_{31}}\right)$$

$$b_{03} = \left[\frac{1}{1-(\alpha+\gamma)}\right]\left[ln(\alpha A) + \gamma ln\left(\frac{\gamma}{\alpha}\right) + \beta ln(h^*)\right]$$
$$= b_{01} - (b_{11} + b_{21} + b_{31})\, ln\,(1 + b_{11} + b_{21} + b_{31}) + (b_{21} + b_{31})\, ln\,(1 + b_{21} + b_{31})$$
$$+b_{11} ln\left(\frac{1+b_{11}}{h^*}\right)$$
$$b_{13} = \frac{\gamma}{1-(\alpha+\gamma)} = b_{11} + b_{21}$$
$$b_{23} = \frac{-1}{1-(\alpha+\gamma)} = b_{31}$$
$$b_{33} = \frac{\gamma-1}{1-(\alpha+\gamma)} = b_{11} + b_{21} + b_{31}$$
$$b_{43} = \frac{g}{1-(\alpha+\gamma)} = b_{41}$$
$$c_{03} = b_{03} + ln\left(\frac{-b_{13}}{1+b_{13}+b_{23}}\right)$$
$$= b_{01} - (b_{11} + b_{21} + b_{31})\, ln\,(1 + b_{11} + b_{21} + b_{31}) + (b_{21} + b_{31})\, ln\,(1 + b_{21} + b_{31})$$
$$+b_{11} ln\left[\frac{1+b_{11}}{h^*}\right]$$
$$c_{13} = 1 + b_{13} = b_{11} + b_{21}$$
$$c_{23} = b_{23} = b_{31}$$
$$c_{33} = -\,(1 + b_{13} + b_{23}) = -\,(1 + b_{11} + b_{21} + b_{31})$$
$$c_{43} = b_{43} = b_{41}$$

We use the indicator variables defined above for each hours regime: $D_{1t} = 1(h_t < h^*), D_{2t} = 1(h_t > h^*)$, and $D_{3t} = 1 - D_{1t} - D_{2t} = 1(h_t = h^*)$. The demand system for the inputs is specified below.

$$ln(h_t) = D_{1t}\left[ln\left(\frac{b_{11}}{1+b_{21}+b_{31}}\right) + ln\left(\frac{B_t}{w_t}\right)\right]$$
$$+\ D_{2t}\left\{ln\left(\frac{b_{11}}{1+b_{21}+b_{31}}\right) + ln\left[(1-\lambda)\,h^* + \left(\frac{B_t}{w_t}\right)\right] - ln(\lambda)\right\}$$
$$+ D_{3t} ln\,(h^*) + \varepsilon_{ht}$$

$$
\begin{aligned}
ln\left(E_t\right) = & D_{1t}\left[b_{01} + b_{11} ln\left(\frac{w_t}{r_t}\right) + b_{21} ln\left(\frac{B_t}{r_t}\right) + b_{31} ln\left(\frac{B_t}{p_t}\right) + b_{41} t\right] \\
& + D_{2t}\left\{b_{01} + (1 - b_{31}) ln\left(\frac{1 + b_{11} + b_{21} + b_{31}}{b_{31}}\right) + b_{11} ln\left[\frac{(1 + b_{11})(\lambda)}{b_{11}}\right] t\right. \\
& \left. + (b_{11} + b_{21})\, ln\left(\frac{w_t}{r_t}\right) + b_{31}\ ln\left(\frac{w_t}{p_t}\right) + (b_{21} + b_{31})\, ln\left[(1 - \lambda)\, h^* + \frac{B_t}{w_t}\right] + b_{41} t\right\} \\
& + D_{3t}\left[b_{01} - (b_{11} + b_{21} + b_{31})\, ln\,(1 + b_{11} + b_{21} + b_{31})\right. \\
& + (b_{21} + b_{31})\, ln\,(1 + b_{21} + b_{31}) + b_{11} ln\left[\frac{1 + b_{11}}{h^*}\right] + (b_{11} + b_{21})\, ln\left(\frac{w_t}{r_t}\right) \\
& \left. + b_{31} ln\left(\frac{w_t}{p_t}\right) + (b_{11} + b_{21} + b_{31})\, ln\left(h^* + \frac{B_t}{w_t}\right) + b_{41} t\right\} + \varepsilon_{Et}
\end{aligned}
$$

$$
\begin{aligned}
ln\left(K_t\right) = & D_{1t}\left\{b_{01} + ln\left[\frac{-(b_{11} + b_{21})}{1 + b_{21} + b_{31}}\right] + b_{11} ln\left(\frac{w_t}{r_t}\right) + (1 + b_{21})\, ln\left(\frac{B_t}{r_t}\right)\right. \\
& \left. + b_{31} ln\left(\frac{B_t}{p_t}\right) + b_{41} t\right\} \\
& = D_{2t}\left\{b_{01} + (1 - b_{31}) ln\left(\frac{1 + b_{11} + b_{21} + b_{31}}{b_{31}}\right) + b_{11} ln\left[\frac{(1 + b_{11})(\lambda)}{b_{11}}\right]\right. \\
& + ln\left[\frac{-(b_{11} + b_{12})}{1 + b_{21} + b_{31}}\right] + (1 + b_{11} + b_{21})\, ln\left(\frac{w_t}{r_t}\right) + b_{31} ln\left(\frac{w_t}{p_t}\right) \\
& \left. (1 + b_{21} + b_{31})\, ln\left[(1 - \lambda)\, h^* + \frac{B_t}{w_t}\right] + b_{41} t\right\} \\
& = D_{3t}\left\{b_{01} - (b_{11} + b_{21} + b_{31})\, ln\,(1 + b_{11} + b_{21} + b_{31}) + (b_{21} + b_{31})\, ln\,(1 + b_{21} + b_{31})\right. \\
& + b_{11} ln\left[\frac{1 + b_{11}}{h^*}\right] + (b_{11} + b_{21})\, ln\left(\frac{w_t}{r_t}\right) + b_{31} ln\left(\frac{w_t}{p_t}\right) \\
& \left. - (1 + b_{11} + b_{21} + b_{31})\, ln\left(h^* + \frac{B}{w}\right) + b_{41} t\right\} + \varepsilon_{Kt}
\end{aligned}
$$

As long as one knows which regime corresponds to each period in the data, the CD production function parameters and the long-run profit maximizing input demand function parameters can be estimated from data on employment alone. Furthermore, if w_t, r_t, B_t, and p_t are exogenous, the NLSUR estimator is consistent. If p_t were treated as endogenous, then a nonlinear 3SLS estimator would be consistent.

Cobb-Douglas Example

	Cost Minimization												Profit Maximization											
	h < h*				h = h*				h > h*				h < h*				h = h*				h > h*			
Variable	h	E	K	H	h	E	K	H	h	E	K	H	h	E	K	H	h	E	K	H	h	E	K	H
W	-	+	+	-	0	-	+	-	-	+	+	-	-	-	-	-	0	-	-	-	-	-	-	-
B	+	-	+	+	0	-	+	-	+	-	+	+	+	-	-	-	0	-	-	-	+	-	-	-
r	0	+	-	+	0	+	-	+	0	+	-	+	0	-	-	-	0	-	-	-	0	-	-	-
Q	0	+	+	+	0	+	+	+	0	+	+	+												
P													0	+	+	+	0	+	+	+	0	+	+	+
γ					0	0	0	0	-	+	+	?					0	0	0	0	-	?	?	?
h*					+	-	-	+	-	+	-	-					+	-	-	?	-	+	+	-