Notes on Empirical Labor Demand Under A Cobb-Douglas Technology

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1 Empirical Regimes

1. Inputs L & K are exogenous.
2. Output Q is exogenous.
3. Output price p is exogenous.
4. Budget C is exogenous.
5. Output Q and output price p are endogenous.

In addition to the time trend $'t'$, it will be assumed that wages w, rental rate r, and consumer income y are exogenous. The estimators described below if not always unbiased are consistent. For convenience it is assumed that we have time-series data, although one could have a cross-section sample of firms in a particular industry or even panel data for a sample of firms in an industry.

2 Aggregation from Firm to Industry Scale

Empirically, one is often confined to using industry level data. One way to rationalize the aggregation of the model to the industry level is to assume that each firm in the industry is a clone of one another. For convenience we will ignore the stochastic element. In this setting we specify the CD technology for the representative firm by

$$q_t = a_t l_t^{\alpha} k_t^{\beta} e^{gt}$$

where a_t is a firm-level scale factor defined as $a_t = AN_t^{-(1-\alpha-\beta)}$, and N_t is the number of firms in the industry. We will treat the number of firms in the industry as exogenously determined.

Scaled up to the industry level, the production technology is derived as follows:

$$N_t q_t = N_t^{(1-\alpha-\beta)} a_t \left(N_t l_t\right)^{\alpha} \left(N_t k_t\right)^{\beta} e^{gt},$$

$\Rightarrow$

$$Q_t = AL_t^{\alpha} K_t^{\beta} e^{gt},$$

where $Q_t = N_t q_t, A = N_t^{(1-\alpha-\beta)} a_t, L_t = (N_t l_t)$, and $K_t = (N_t k_t)$. If one could observe N_t, it would be possible to express the production function in firm per-capita terms for estimation purposes though there would be no particular advantage for doing so:

$$q_t = AN_t^{-(1-\alpha-\beta)} l_t^{\alpha} k_t^{\beta} e^{gt}.$$

3 Special Elasticity of Substitution Case

The stochastic capital/labor ratio form of the model is described by

$$ln\left(\frac{K_t}{L_t}\right) - ln\left(\frac{w_t}{r_t}\right) = ln\left(\frac{\beta}{\alpha}\right) + u_{kt} - u_{lt},$$

$\Rightarrow$

$$ln\left(\frac{r_t K_t}{w_t L_t}\right) = b + u_{klt}$$

where $b = ln\left(\frac{\beta}{\alpha}\right)$ and $u_{klt} = u_{kt} - u_{lt}$. This specification imposes the restriction that the elasticity of substitution $\sigma = 1$. To estimate this model one does not require data on Q. All that is required are data on the cost outlays for labor and the nonlabor inputs. As we will see below, in the case of a public agency with an exogenous budget all that is needed are data on the budget share of one of the inputs. While this CES form of the model can be estimated by OLS, the production function parameters α, β, g, and A are not identified. All that can be estimated is $\hat{b} = \widehat{ln\left(\frac{\beta}{\alpha}\right)}$ or $\widehat{\left(\frac{\beta}{\alpha}\right)} = e^{\hat{b}}$.

4 Cobb-Douglas Production Function With Exogenous Inputs

$$Q_t = Ae^{gt} L_t^{\alpha} K_t^{\beta} e^{u_{st}}$$

or

$$ln(Q_t) = ln(A) + gt + \alpha ln(L_t) + \beta ln(K_t) + u_{st}$$

where u_{st} is a mean zero disturbance term and $g > 0, 0 < \alpha, \beta < 1$, and $0 < \alpha + \beta < 1$. If u_{st} is i.i.d., then the log production function could be estimated by OLS. Exogenous L and K, is not a very likely scenario but is in principle possible, e.g. some legal or contractual requirement that dictates the usage of L and K. To estimate this model, we require data on Q_t, L_t, and K_t.

5 Exogenous Output

Conditional stochastic input demand functions can be rationalized by exogenous output (production orders) determined by corporate headquarters of a multi-branch firm. The plant manager's objective is to produce a given level of output at the lowest cost. Empirically, the stochastic conditional input demand functions can be expressed as

$$ln(L_t) = b_0 + b_1 ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{lt}$$
$$ln(K_t) = a_0 + a_1 ln\left(\frac{w_t}{r_t}\right) + a_2 ln(Q_t) + a_3 t + u_{kt},$$

where u_{lt} and u_{kt} are mean zero error terms. If the error terms are i.i.d., one can estimate the conditional input demand functions by OLS. In light of the CD model, this estimation strategy would be unbiased and consistent but inefficient because of implied cross-equation restrictions on the parameters and likely covariance between u_{lt} and u_{kt}.

The implied parameter restrictions are both within equations and across equations as seen below:

$b_0 = \left(\frac{-1}{\alpha+\beta}\right) ln \left[A\left(\frac{\beta}{\alpha}\right)^{\beta}\right]$	$a_0 = \left(\frac{-1}{\alpha+\beta}\right) ln \left[A\left(\frac{\alpha}{\beta}\right)^{\alpha}\right] = b_0 + ln\left(\frac{-b_1}{1+b_1}\right)$
$b_1 = \frac{-\beta}{\alpha+\beta} < 0$	$a_1 = \frac{\alpha}{\alpha+\beta} = 1 + b_1 > 0$
$b_2 = \frac{1}{\alpha+\beta} > 0$	$a_2 = \frac{1}{\alpha+\beta} = b_2 > 0$
$b_3 = \frac{-g}{\alpha+\beta} < 0$	$a_3 = \frac{-g}{\alpha+\beta} = b_3 < 0$

The production function parameters can be recovered from the conditional input demand function parameters:

$$\alpha = \frac{1+b_1}{b_2} = \frac{a_1}{a_2}$$

$$\beta = \frac{-b_1}{b_2} = \frac{1-a_1}{a_2}$$

$$g = \frac{-b_3}{b_2} = \frac{-a_3}{a_2}$$

$$ln(A) = \left(\frac{1}{b_2}\right)\left[b_1 ln\left(\frac{-b_1}{1+b_1}\right) - b_0\right] = \left(\frac{1}{a_2}\right)\left[a_0 - a_1 ln\left(\frac{1-a_1}{a_1}\right)\right]$$

$$A = exp\left\{\left(\frac{1}{b_2}\right)\left[b_1 ln\left(\frac{-b_1}{1+b_1}\right) - b_0\right]\right\} = exp\left\{\left(\frac{1}{a_2}\right)\left[a_0 - a_1 ln\left(\frac{1-a_1}{a_1}\right)\right]\right\}.$$

Suppose

$$\begin{pmatrix} u_{lt} \\ u_{kt} \end{pmatrix} \sim N \begin{pmatrix} \sigma_{ll} & \sigma_{lk} \\ \sigma_{kl} & \sigma_{kk} \end{pmatrix}.$$

The data requirements for efficient (joint) estimation of the conditional input demand functions are that data must be available on $L_t, K_t, Q_t, w_t,$ and r_t. Estimation of the demand function parameters with cross-equation restrictions by Non-Linear Seemingly Unrelated Regression (NLSUR) is applied to

$$ln(L_t) = b_0 + b_1 ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{lt}$$
$$ln(K_t) = b_0 + ln\left(\frac{-b_1}{1+b_1}\right) + (1+b_1)\, ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{kt}.$$

If we let $ln\,(Z_t) = ln(K_t) - ln\left(\frac{w_t}{r_t}\right) = ln\left(\frac{r_t K_t}{w_t}\right)$, then an equivalent specification of the model is given by

$$ln(L_t) = b_0 + b_1 ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{lt} \quad (1)$$
$$ln(Z_t) = b_0 + ln\left(\frac{-b_1}{1+b_1}\right) + b_1 ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{kt}. \quad (2)$$

In this case we could estimate both the b and the a parameters of the conditional input demand functions along with the standard errors using non-linear seemingly unrelated regression. Starting values of the parameters can be obtained from OLS estimation of equation (1). The estimated conditional input demand function parameters can be used to recover the production function parameters:

$$\tilde{A} = exp\left\{\left(\frac{1}{\hat{b}_2^{nlsur}}\right)\left[\hat{b}_1^{nlsur} ln\left(\frac{-\hat{b}_1^{nlsur}}{1+\hat{b}_1^{nlsur}}\right) - \hat{b}_0^{nlsur}\right]\right\}$$

$$\tilde{\alpha} = \left(\frac{1+\hat{b}_1^{nlsur}}{\hat{b}_2^{nlsur}}\right)$$

$$\tilde{\beta} = \frac{-\hat{b}_1^{nlsur}}{\hat{b}_2^{nlsur}}$$

$$\tilde{g} = \frac{-\hat{b}_3^{nlsur}}{\hat{b}_2^{nlsur}}.$$

In this case we would have to separately estimate the standard errors of the derived production function parameters.

One could use NLSUR to estimate the production function parameters directly and then back out the conditional input demand function parameters:

$$ln(L_t) = \left(\frac{-1}{\alpha+\beta}\right)\left\{\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha}\right)\right] + \beta ln\left(\frac{w_t}{r_t}\right) + gt - ln(Q_t)\right\} + u_{lt} \tag{3}$$

$$ln(K_t) = \left(\frac{-1}{\alpha+\beta}\right)\left\{\left[ln(A) - \alpha ln\left(\frac{\beta}{\alpha}\right)\right] - \alpha ln\left(\frac{w_t}{r_t}\right) + gt - ln(Q_t)\right\} + u_{kt}. \tag{4}$$

The implied estimates of the conditional input demand function parameters are derived from

$$\boxed{\begin{array}{ll}
\tilde{b}_0 = \dfrac{-\left[ln(\hat{A}^{nlsur}) + \hat{\beta}^{nlsur} ln\left(\dfrac{\hat{\beta}^{nlsur}}{\hat{\alpha}^{nlsur}}\right)\right]}{\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}}, & \tilde{a}_0 = \tilde{b}_0 + ln\left(\dfrac{-\tilde{b}_1}{1+\tilde{b}_1}\right) \\
\tilde{b}_1 = \dfrac{-\hat{\beta}^{nlsur}}{\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}} < 0, & \tilde{a}_1 = 1 + \tilde{b}_1 > 0 \\
\tilde{b}_2 = \dfrac{1}{\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}} > 0, & \tilde{a}_2 = \tilde{b}_2 > 0 \\
\tilde{b}_3 = \dfrac{-\hat{g}^{nlsur}}{\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}} < 0, & \tilde{a}_3 = \tilde{b}_3 < 0.
\end{array}}$$

In this case we would have to separately estimate the standard errors of the derived estimates of the conditional input demand function parameters.

How does NLSUR work? The NLSUR method can improve efficiency when error terms are correlated across a system of equations and when there are across-equation restrictions on the parameters. A system of M nonlinear equations with additive errors are represented as

$$
\begin{aligned}
y_1 &= h_1(\beta, X) + u_1 \\
&. \\
&. \\
&. \\
y_M &= h_M(\beta, X) + u_{M,}
\end{aligned}
$$

where the $y^{'s}$ and the $u^{'s}$ are T x 1 vectors, β is a parameter vector, and X is the observations on all of the variables. Note that not all elements of β and not all variables in X need appear in every equation. It is assumed that the errors are all zero mean, are uncorrelated with X, and have a M x M contemporaneous variance-covariance matrix:

$$
\begin{aligned}
\Sigma &= E(u_t^{'} u_t) \\
&= \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \cdot\;\cdot\;\cdot & \sigma_{1M} \\
\sigma_{21} & \sigma_{22} & \cdot\;\cdot\;\cdot & \sigma_{2M} \\
. & . & & . \\
. & . & & . \\
. & . & & . \\
\sigma_{M1} & \sigma_{M2} & \cdot\;\cdot\;\cdot & \sigma_{MM}
\end{pmatrix},
\end{aligned}
$$

where u_t is the 1 x M vector of observations on all of the equation disturbances in period t, i.e. $u_t = (u_{t1}, ..., u_{tM})$.

Ideally, the GLS estimator would be used to minimize the generalized sum of squared residuals

$$u(\beta)' \left(\Sigma^{-1} \otimes \mathrm{I}_T\right) u(\beta) = \sum_{i=1}^{M} \sum_{j=1}^{M} \sigma^{ij} \left[y_i - h_i\left(\beta, X\right)\right]' \left[y_j - h_j\left(\beta, X\right)\right],$$

where $u(\beta)$ is the MT x 1 vector of stacked error terms, σ^{ij} is the ijth element of Σ^{-1}, $y_i - h_i\left(\beta, X\right) = u_i\left(\beta\right)$ and $y_j - h_j\left(\beta, X\right) = u_j\left(\beta\right)$ are the T x 1 error term vectors corresponding to the ith and the jth equations. The F.O.C. are represented by

$$\frac{\partial \left[u(\beta)' \left(\Sigma^{-1} \otimes \mathrm{I}_T\right) u(\beta)\right]}{\partial \beta} = \sum_{i=1}^{M} \sum_{j=1}^{M} \sigma^{ij} \left[2H_i\left(\beta\right)' u_j\left(\beta\right)\right] = 0,$$

where $H_i\left(\beta\right)$ is the T x K observation matrix obtained from $\dfrac{\partial h_i(X, \beta)}{\partial \beta}$. This gives rise to a set of K nonlinear equations in K unknowns. Note: for any parameter in β that does not appear in the ith equation, the corresponding column of $H_i\left(\beta\right)$ will be a column of 0's.

Since the elements of Σ are unknown, these have to be estimated. One could first estimate each equation by nonlinear least squares (NLS) to obtain $\hat{u}_i = y_i - h_i\left(X, \hat{\beta}^{nls}\right)$ and then estimate $\hat{\sigma}_{ij} = \dfrac{\hat{u}_i' \hat{u}_j}{T}$ which in turn is used to fill out the elements of $\hat{\Sigma}$. The next step is to use nonlinear optimization methods to estimate $\hat{\beta}^{(1)}$ as the solution to

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \hat{\sigma}^{ij} \left[2H_i\left(\hat{\beta}^{(1)}\right)' u_j\left(\hat{\beta}^{(1)}\right)\right] = 0.$$

The process can be iterated to obtain a new set of values for $\hat{u}_i$, $\hat{\Sigma}$, and $\hat{\beta}$ until convergence. The asymptotic variance-covariance matrix is estimated by

$$\widehat{Var}\left(\hat{\beta}^{nlsur}\right) = \sum_{i=1}^{M} \sum_{j=1}^{M} \hat{\sigma}^{ij(nlsur)} \left[H_i^{nlsur}(\hat{\beta}^{nlsur})\right]' \left[H_j^{nlsur}(\hat{\beta}^{nlsur})\right].$$

Suppose that no data are available on K_t. As long as data are available on Q_t, L_t, w_t, and r_t, one can estimate the conditional input demand function for labor and recover all of the conditional input demand function parameters and production function parameters. Consider OLS direct estimation of the conditional input demand function for labor:

$$ln(L_t) = b_0 + b_1 ln\left(\frac{w_t}{r_t}\right) + b_2 ln(Q_t) + b_3 t + u_{lt}.$$

As shown below, recovery of the parameters for the conditional input demand function for capital/nonlabor is straightforward:

$\tilde{a}_0 = \hat{b}_0^{ols} + ln\left(\frac{-\hat{b}_1^{ols}}{1+\hat{b}_1^{ols}}\right)$
$\tilde{a}_1 = 1 + \hat{b}_1^{ols} > 0$
$\tilde{a}_2 = \hat{b}_2^{ols} > 0$
$\tilde{a}_3 = \hat{b}_3^{ols} < 0.$

Because of the linear relationships between the estimators $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ and the estimators $\hat{b}_1^{ols}, \hat{b}_2^{ols}, \hat{b}_3^{ols}$, the estimated standard errors are identical between these two sets of estimators. However, the estimated standard error on the constant term $\tilde{a}_0$ is a bit more complicated because of the nonlinear relationship with the estimator $\hat{b}_1^{ols}$. Therefore, the standard error for $\tilde{a}_0$ has to be estimated separately.

It should be clear that the underlying production function parameters are identified from OLS estimation of the conditional input demand function for labor. This is seen below.

$$\tilde{A} = exp\left\{\left(\frac{1}{\hat{b}_2^{ols}}\right)\left[\hat{b}_1^{ols} ln\left(\frac{-\hat{b}_1^{ols}}{1+\hat{b}_1^{ols}}\right) - \hat{b}_0^{ols}\right]\right\}$$

$$\tilde{\alpha} = \left(\frac{1+\hat{b}_1^{ols}}{\hat{b}_2^{ols}}\right)$$

$$\tilde{\beta} = \frac{-\hat{b}_1^{ols}}{\hat{b}_2^{ols}}$$

$$\tilde{g} = \frac{-\hat{b}_3^{ols}}{\hat{b}_2^{ols}}.$$

Of course it will be necessary to separately obtain the standard errors on the estimated production parameters derived from OLS estimation of the conditional input demand function for labor.

If one is interested in estimating the production function parameters directly from the conditional input demand function for labor, one would use Non-Linear Least Squares (NLS) to estimate

$$ln(L_t) = \left(\frac{-1}{\alpha+\beta}\right)\left\{\left[ln(A) + \beta ln\left(\frac{\beta}{\alpha}\right)\right] + \beta ln\left(\frac{w_t}{r_t}\right) + gt - ln(Q_t)\right\} + u_{lt}.$$

From the NLS estimates of the conditional labor input demand function, one can recover the direct conditional input demand function parameters for both labor and the nonlabor inputs:

$\tilde{b}_0 = \dfrac{-\left[ln(\hat{A}^{nls}) + \hat{\beta}^{nls} ln\left(\dfrac{\hat{\beta}^{nls}}{\hat{\alpha}^{nls}}\right)\right]}{\hat{\alpha}^{nls} + \hat{\beta}^{nls}}$	$\tilde{a}_0 = \tilde{b}_0 + ln\left(\dfrac{-\tilde{b}_1}{1+\tilde{b}_1}\right)$
$\tilde{b}_1 = \dfrac{-\hat{\beta}^{nls}}{\hat{\alpha}^{nls} + \hat{\beta}^{nls}} < 0$	$\tilde{a}_1 = 1 + \tilde{b}_1 > 0$
$\tilde{b}_2 = \dfrac{1}{\hat{\alpha}^{nls} + \hat{\beta}^{nls}} > 0$	$\tilde{a}_2 = \tilde{b}_2 > 0$
$\tilde{b}_3 = \dfrac{-\hat{g}^{nls}}{\hat{\alpha}^{nls} + \hat{\beta}^{nls}} < 0$	$\tilde{a}_3 = \tilde{b}_3 < 0.$

Again, the standard errors of the derived parameters need to be separately estimated.

How does NLS work? Imagine an additive error regression model of the form

$$Y_t = h(X_t, \beta) + u_t,$$

where β is a K x 1 parameter vector to be estimated, $Cov\left[u_t, h(X_t, \beta)\right] = 0$, and u_t is i.i.d. with mean zero and constant variance. In particular consider models in which $h(\cdot)$ is not a linear function of the parameters.

NLS minimizes the sum of squared residuals:

$$S(\hat{\beta}^{nls}) = \left(\frac{1}{2}\right) \sum_{t=1}^{T} \left[Y_t - h(X_t, \hat{\beta}^{nls})\right]^2,$$

where $\left(\dfrac{1}{2}\right)$ is scaling factor that does not affect the solution. The corresponding F.O.C. are

$$\frac{\partial S(\hat{\beta}^{nls})}{\partial \hat{\beta}^{nls}} = -\sum_{t=1}^{T} \left[Y_t - h(X_t, \hat{\beta}^{nls})\right] \frac{\partial h(X_t, \hat{\beta}^{nls})}{\partial \hat{\beta}^{nls}} = 0$$

which produces a set of nonlinear normal equations that have to be solved by nonlinear optimization routines. The Gauss-Newton method is a very common optimization routine that is used for NLS. Essentially, this method iteratively minimizes the sums of squared residuals from linear approximations to the model.

A linear approximation to the nonlinear model follows from a first-order Taylor series approximation to $h(X_t, \beta)$ evaluated at some particular parameter vector value β^0:

$$h(X_t, \beta) \approx h(X_t, \beta^0) + \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \left(\beta_k - \beta_k^0\right).$$

Therefore,

$$h(X_t, \beta) = h(X_t, \beta^0) + \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \left(\beta_k - \beta_k^0\right) + R_t^0,$$

where R_t^0 is the remainder term, i.e.

$$R_t^0 = h(X_t, \beta) - h(X_t, \beta^0) - \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \left(\beta_k - \beta_k^0\right).$$

Notice that we can express this linearized model as a conventional regression model:

$$Y_t = h(X_t, \beta^0) + \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \left(\beta_k - \beta_k^0\right) + R_t^0 + u_t$$

$\Rightarrow$

$$Y_t - h(X_t, \beta^0) + \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \beta_k^0 = \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \beta_k + R_t^0 + u_t$$

or

$$Y_t^0 = \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \beta_k + u_t^0,$$

where $Y_t^0 = Y_t - h(X_t, \beta^0) + \sum_{k=1}^{K} \frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0} \beta_k^0$, $u_t^0 = R_t^0 + u_t$, and $\frac{\partial h(X_t, \beta^0)}{\partial \beta_k^0}$ is the kth regressor. We can express the linearized regression model in matrix form:

$$Y^0 = H^0 \beta + u^0, \tag{5}$$

where Y^0 and u^0 are T x 1 observation vectors corresponding to Y_t^0 and u_t^0, and H^0 is the T x K observation matrix corresponding to the K regressors given by $\dfrac{\partial h(X_t, \beta^0)}{\partial \beta_k^0}$, $k = 1, ..., K$. Our first round estimate of β is obtained from

$$\hat{\beta}^{(1)} = \left(H^{0\prime} H^0\right)^{-1} H^{0\prime} Y^0.$$

Continuing onto the next iteration, we run the regression

$$Y^{(1)} = H^{(1)}\beta + u^{(1)},$$

where the elements of $Y^{(1)}$ are obtained from $Y_t^{(1)} = Y_t - h(X_t, \hat{\beta}^{(1)}) + \sum_{k=1}^{K} \dfrac{\partial h(X_t, \hat{\beta}^{(1)})}{\partial \beta_k^{(1)}} \beta_k^{(1)}$ and $H^{(1)}$ is the T x K observation matrix corresponding to the K regressors given by $\dfrac{\partial h(X_t, \hat{\beta}^{(1)})}{\partial \beta_k^{(1)}}$, $k = 1, ..., K$. The second round estimate of β is obtained from

$$\hat{\beta}^{(2)} = \left(H^{(1)\prime} H^{(1)}\right)^{-1} H^{(1)\prime} Y^{(1)}.$$

Subsequent iterations occur until $\hat{\beta}^{(j)} - \hat{\beta}^{(j-1)} \approx 0$ or $\dfrac{\hat{\beta}^{(j)} - \hat{\beta}^{(j-1)}}{\hat{\beta}^{(j-1)}} \approx 0$. If convergence is reached on the jth iteration, then $\hat{\beta}^{nls} = \hat{\beta}^{(j)}$. The estimated variance/covariance matrix for the NLS estimator is given by

$$\widehat{Var}(\hat{\beta}^{nls}) = \hat{\sigma}^2 \left(H' H\right)^{-1}$$

where $\hat{\sigma}^2 = \dfrac{\hat{u}^{(nls)\prime} \hat{u}^{(nls)}}{T \text{ or } T - K}$, $\hat{u}^{(nls)} = Y_t - h(X_t, \hat{\beta}^{nls})$, and $H = H^{(j)}$ or $H^{(j-1)}$.

An equivalent way to express the Gauss-Newton non-linear method is to subtract $H^0\beta^0$ from both sides of (5) to obtain

$$\Delta Y^0 = H^0 \Delta\beta^{(1)} + u^0 \quad (6)$$

where $\Delta Y^0 = Y^0 - H^0\beta^0$,i.e. $\Delta Y_t^0 = Y_t - h(X_t, \beta^0)$, and $\Delta\beta^{\ (1)} = (\beta - \beta^0)$. Now we could estimate (6) by OLS to obtain

$$\widehat{\Delta\beta}^{(1)} = \left(H^{0\prime}H^0\right)^{-1} H^{0\prime}\Delta Y^0.$$

Next, we back out our 1st round estimate of β from

$$\hat{\beta}^{(1)} = \widehat{\Delta\beta}^{(1)} + \beta^0.$$

Continuing onto the next iteration, we run the regression

$$\Delta Y^{(1)} = H^{(1)}\Delta\beta^{(2)} + u^{(1)},$$

where $\Delta Y_t^{(1)} = Y_t - h(X_t, \hat{\beta}^{(1)})$ and $H^{(1)}$ is the T x K observation matrix corresponding to the K regressors given by $\dfrac{\partial h(X_t, \hat{\beta}^{(1)})}{\partial \hat{\beta}_k^{(1)}}$. The second round estimate of β is obtained from

$$\hat{\beta}^{(2)} = \widehat{\Delta\beta}^{(2)} + \hat{\beta}^{(1)}.$$

One can see that this is an iterative process, so on the jth iteration we have an updated estimate

$$\hat{\beta}^{(j)} = \widehat{\Delta\beta}^{(j)} + \hat{\beta}^{(j-1)}.$$

Convergence is reached when $\widehat{\Delta\beta}^{(j)} \approx 0$ according to some numerical criterion, e.g. $\widehat{\Delta\beta}^{(j)} < 10^{-5}$.

The estimated asymptotic variance covariance matrix for $\hat{\beta}^{nls}$ is estimated by

$$\widehat{Var}\left(\hat{\beta}^{nls}\right) = \hat{\sigma}^2\left(H^{nls\prime}H^{nls}\right)^{-1},$$

where $\hat{\sigma}^2 = \dfrac{\sum_{t=1}^{T}\left[Y_t - h(X_t, \hat{\beta}^{nls})\right]^2}{T-K \text{ or } T}$ and H is the T x K observation matrix corresponding to the K regressors given by $\dfrac{\partial h(X_t, \hat{\beta}^{nls})}{\partial \hat{\beta}_k^{nls}}$.

How does one estimate separately the standard errors of derived parameter estimators? It is fairly common to encounter situations in which the parameters of interest are nonlinear functions of a basic set of parameters that can be more directly estimated. The Cobb-Douglas case is the example we have examined above. One can derive estimates of the parameters of interest from estimates of the basic set of parameters. Obtaining estimated standard errors for the estimated parameters of interest can be a bit more involved. Examples of this have already been shown with several nonlinear estimation techniques. If one does not use a nonlinear estimator program, the delta method can be used to manually obtain estimated asymptotic standard errors for the parameters of interest.

Basically, the **delta method** is a first-order Taylor series approximation to the nonlinear relationship between the parameters of interest and a basic set of parameters. As an example, suppose our parameter of interest is δ and it is related to a set of basic parameters $\theta = (\theta_1, ..., \theta_K)$ via the nonlinear relationship $\delta = h(\theta_1, ..., \theta_K)$. We could obtain our estimator of δ as

$$\tilde{\delta} = h\left(\hat{\theta}_1, ..., \hat{\theta}_K\right). \tag{7}$$

Our first-order Taylor series expansion for (7) is given by

$$\tilde{\delta} = h(\theta_1, ..., \theta_K) + \sum_{k=1}^{K}\left(\frac{\partial h}{\partial \theta_k}\right)\left(\hat{\theta}_k - \theta_k\right) + R_\delta,$$

where R_δ is a remainder term that is assumed asymptotically to approach 0 as the sample size approaches infinity. The asymptotic variance of $\tilde{\delta}$ is accordingly expressed as

$$
\begin{aligned}
Var\left(\tilde{\delta}\right) &= \sum_{k=1}^{K}\left(\frac{\partial h}{\partial \theta_k}\right)^2 Var\left(\hat{\theta}_k\right) + 2\sum_{j<k}\sum_{k=2}^{K}\left(\frac{\partial h}{\partial \theta_j}\right)\left(\frac{\partial h}{\partial \theta_k}\right) Cov\left(\hat{\theta}_j, \hat{\theta}_k\right) \\
&= \left(\frac{\partial h}{\partial \theta}\right)' \Sigma_{\hat{\theta}} \left(\frac{\partial h}{\partial \theta}\right),
\end{aligned}
$$

where $\Sigma_{\hat{\theta}} = Var(\hat{\theta})$ is the K x K variance-covariance matrix for the estimator $\hat{\theta}$, and $\left(\frac{\partial h}{\partial \theta}\right)'$ is the 1xK gradient vector $\left(\frac{\partial h}{\partial \theta_1} \quad \cdot \quad \cdot \quad \cdot \quad \frac{\partial h}{\partial \theta_K}\right)$. In practice estimated values are substituted for the unknown true parameter values:

$$
\widetilde{Var}\left(\tilde{\delta}\right) = \left(\frac{\partial h}{\partial \theta}|_{\hat{\theta}}\right)' \hat{\Sigma}_{\hat{\theta}} \left(\frac{\partial h}{\partial \theta}|_{\hat{\theta}}\right).
$$

The asymptotic standard error for $\tilde{\delta}$ would be estimated as $\tilde{\sigma}_{\tilde{\delta}} = \sqrt{\widetilde{Var}\left(\tilde{\delta}\right)}$

A simple application of the delta method is obtaining the estimated standard error for the Cobb-Douglas production function parameter β derived from the conditional input demand function for labor,

$$
\begin{aligned}
\beta &= h(b_1, b_2) \\
&= \frac{-b_1}{b_2}.
\end{aligned}
$$

In this case $\frac{\partial h}{\partial b_1} = \frac{-1}{b_2}$ and $\frac{\partial h}{\partial b_2} = \frac{b_1}{(b_2)^2}$ so that

$$
\widetilde{Var}\left(\tilde{\beta}\right) = \left(\frac{-1}{\hat{b}_2}\right)^2 \widehat{Var}\left(\hat{b}_1\right) + \left(\frac{\hat{b}_1}{\left(\hat{b}_2\right)^2}\right)^2 \widehat{Var}\left(\hat{b}_2\right) + 2\left(\frac{-1}{\hat{b}_2}\right)\left(\frac{\hat{b}_1}{\left(\hat{b}_2\right)^2}\right) \widehat{Cov}\left(\hat{b}_1, \hat{b}_2\right).
$$

Another method for obtaining estimated standard errors is the **bootstrapping** technique. This technique is particularly useful if the underlying model is too complicated for the delta method to be practical. Suppose a parameter vector θ is estimated from a sample of size T described by $Z = [w_1, ..., w_T]$. Denote this estimator by $\hat{\theta}_T$.

Suppose we treat the sample Z as the population and accordingly draw B samples of size T with replacement from the empirical distribution for Z. From these B replications we can obtain B estimates of θ: $\left\{\hat{\theta}_T^b \mid b = 1, ..., B\right\}$. The estimated asymptotic variance-covariance matrix for $\hat{\theta}_T$ is obtained as

$$\widetilde{Var}\left(\hat{\theta}_T\right) = \left(\frac{1}{B-1}\right)\sum_{b=1}^{B}\left[\hat{\theta}_T^b - \overline{\hat{\theta}}\right]\left[\hat{\theta}_T^b - \overline{\hat{\theta}}\right]', \tag{8}$$

where $\overline{\hat{\theta}} = \left(\frac{\sum_{b=1}^{B}\hat{\theta}_T^b}{B}\right)$. The estimated asymptotic standard errors are the square roots of the diagonal elements of $\widetilde{Var}\left(\hat{\theta}_T\right)$. There is no hard and fast rule about the number of replications but the default in STATA is $B = 50$. This particular bootstrapping method is **nonparametric** as it uses the empirical distribution for the sample. If one is working with a regression model, then $w_t = (Y_t, X_t)$ and this is an example of a **paired bootstrap** since both Y_t and X_t are resampled.

A **parametric** bootstrap method would be one in which we have a priori knowledge of the distribution of $Y|X$. Suppose we believe $Y|X \sim F(X, \theta)$. One could use the original sample values for X, i.e. $(X_1, ..., X_T)$, and generate repeated samples (with replacement) on Y from $F(X, \hat{\theta})$. This corresponds to treating the X'^s as fixed. Another **parametric** technique is to treat the $X's$ as stochastic and construct repeated samples (with replacement) for the $X's$ based on their empirical distribution. Next, generate the corresponding values of Y^b from $F(X^b, \hat{\theta})$, $b = 1, ..., B$. We would then obtain B estimates of θ from the sample set $\left\{\left(Y^b, X^b\right) \mid b = 1, ..., B\right\}$.

Consider a regression model with additive i.i.d. errors that do not depend on the model parameters:

$$Y_t = h(X_t, \beta) + u_t, \ t = 1, ..., T.$$

After estimating the model, form the residuals from $\hat{u}_t = Y_t - h(X_t, \hat{\beta})$. We could then bootstrap the residuals, i.e. generate B replication samples (with replacement) on the $\hat{u}'s$ from the empirical distribution of the original sample residuals or from some assumed distribution of the $u's$. We can then bootstrap $Y_t^b = h(X_t, \hat{\beta}) + \hat{u}_t^b$, $b = 1, .., B, t = 1, ..., T$. From the replicated samples $\left(Y^b, X\right)$ we obtain B estimates of the β parameters and apply equation (8) to obtain the bootstrapped standard errors for $\hat{\beta}$. This method is sometimes called the **residual bootstrap**.

6 Exogenous Output Price

Input demand functions obtained under long-run profit maximization with decreasing returns to scale and exogenous output prices can be directly estimated:

$$ln(L_t) = d_o + d_1 ln\left(\frac{w_t}{p_t}\right) + d_2 ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{lt}$$
$$ln(K_t) = c_o + c_1 ln\left(\frac{w_t}{p_t}\right) + c_2 ln\left(\frac{r_t}{p_t}\right) + c_3 t + u_{kt}.$$

If the error terms are i.i.d., one can estimate the long-run input demand functions by OLS. In light of the CD model, this estimation strategy would be unbiased and consistent but inefficient because of implied cross-equation restrictions on the parameters and likely covariance between u_{lt} and u_{kt}.

The implied parameter restrictions are both within equations and across equations as seen below:

$d_0 = \frac{ln(A) + (1-\beta)\, ln\,(\alpha) + \beta ln\,(\beta)}{1-(\alpha+\beta)}$	$c_0 = \frac{ln(A) + (1-\alpha)\, ln\,(\beta) + \alpha ln\,(\alpha)}{1-(\alpha+\beta)} = d_0 + ln\left(\frac{d_2}{1+d_1}\right)$
$d_1 = \frac{-(1-\beta)}{1-(\alpha+\beta)} < 0$	$c_1 = \frac{-\alpha}{1-(\alpha+\beta)} = 1 + d_1 < 0$
$d_2 = \frac{-\beta}{1-(\alpha+\beta)} < 0$	$c_2 = \frac{-(1-\alpha)}{1-(\alpha+\beta)} = d_2 - 1 < 0$
$d_3 = \frac{g}{1-(\alpha+\beta)} > 0$	$c_3 = \frac{g}{1-(\alpha+\beta)} = d_3 > 0$

The production function parameters can be recovered from the long-run input demand function parameters:

$$\alpha = \frac{1+d_1}{d_1+d_2} = \frac{c_1}{c_1+c_2}$$

$$\beta = \frac{d_2}{d_1+d_2} = \frac{1+c_2}{c_1+c_2}$$

$$g = \frac{-d_3}{d_1+d_2} = \frac{-c_3}{d_1+d_2}$$

$$ln(A) = \left(\frac{-1}{d_1+d_2}\right)\left[d_0 + d_1 ln\left(\frac{1+d_1}{d_1+d_2}\right) + d_2 ln\left(\frac{d_2}{d_1+d_2}\right)\right]$$

$$= \left(\frac{-1}{c_1+c_2}\right)\left[c_0 + c_2 ln\left(\frac{1+c_2}{c_1+c_2}\right) + \frac{c_1}{c_1+c_2} ln\left(\frac{c_1}{c_1+c_2}\right)\right]$$

$$A = exp\left\{\left(\frac{-1}{d_1+d_2}\right)\left[d_0 + d_1 ln\left(\frac{1+d_1}{d_1+d_2}\right) + d_2 ln\left(\frac{d_2}{d_1+d_2}\right)\right]\right\}$$

$$= exp\left\{\left(\frac{-1}{c_1+c_2}\right)\left[c_0 + c_2 ln\left(\frac{1+c_2}{c_1+c_2}\right) + \frac{c_1}{c_1+c_2} ln\left(\frac{c_1}{c_1+c_2}\right)\right]\right\}.$$

Again suppose

$$\begin{pmatrix} u_{lt} \\ u_{kt} \end{pmatrix} \sim N \begin{pmatrix} \sigma_{ll} & \sigma_{lk} \\ \sigma_{kl} & \sigma_{kk} \end{pmatrix}.$$

The data requirements for efficient (joint) estimation of the conditional input demand functions are that data must be available on $L_t, K_t, p_t, w_t,$ and r_t. Estimation of the demand function parameters with cross-equation restrictions is applied to

$$ln(L_t) = d_0 + d_1 ln\left(\frac{w_t}{p_t}\right) + d_2 ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{lt}$$
$$ln(K_t) = \left[d_0 + ln\left(\frac{d_2}{1+d_1}\right)\right] + (1+d_1)\, ln\left(\frac{w_t}{p_t}\right) + (d_2 - 1)\, ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{kt}.$$

If we let $ln\,(Z_t) = ln(K_t) - ln\left(\frac{w_t}{p_t}\right) + ln\left(\frac{r_t}{p_t}\right) = ln\left(\frac{r_t K_t}{w_t}\right)$, then an equivalent specification of the restricted model is given by

$$ln(L_t) = d_0 + d_1 ln\left(\frac{w_t}{p_t}\right) + d_2 ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{lt} \tag{9}$$

$$ln(Z_t) = \left[d_0 + ln\left(\frac{d_2}{1+d_1}\right)\right] + d_1 ln\left(\frac{w_t}{p_t}\right) + d_2 ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{kt}. \tag{10}$$

In this case we could estimate both the d and the c parameters of the input demand functions along with the standard errors using NLSUR. Starting values of the parameters can be obtained from OLS estimation of equation (9). The estimated long-run input demand function parameters can be used to recover the production function parameters:

$$\tilde{A} = exp\left\{\left(\frac{-1}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}\right)\left[\hat{d}_0^{nlsur} + \hat{d}_1^{nlsur} ln\left(\frac{1+\hat{d}_1^{nlsur}}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}\right) + \hat{d}_2^{nlsur} ln\left(\frac{\hat{d}_2^{nlsur}}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}\right)\right]\right\}$$

$$\tilde{\alpha} = \frac{1+\hat{d}_1^{nlsur}}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}$$

$$\tilde{\beta} = \frac{\hat{d}_2^{nlsur}}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}$$

$$\tilde{g} = \frac{-\hat{d}_3^{nlsur}}{\hat{d}_1^{nlsur} + \hat{d}_2^{nlsur}}.$$

In this case we would have to separately estimate the standard errors of the derived production function parameters.

One could use NLSUR to estimate the production function parameters directly and then back out the long-run input demand function parameters:

$$ln(L_t) = \left(\frac{-1}{1-\alpha-\beta}\right)\left\{\left[-ln(\alpha A) - \beta ln\left(\frac{\beta}{\alpha}\right)\right] + (1-\beta)\, ln\left(\frac{w_t}{p_t}\right) + \beta ln\left(\frac{r_t}{p_t}\right) - gt\right\} + u_{lt}$$

$$ln(K_t) = \left(\frac{-1}{1-\alpha-\beta}\right)\left\{\left[-ln(\beta A) + \alpha ln\left(\frac{\beta}{\alpha}\right)\right] + \alpha ln\left(\frac{w_t}{p_t}\right) + (1-\alpha)\, ln\left(\frac{r_t}{p_t}\right) - gt\right\} + u_{kt}.$$

The implied estimates of the long-run input demand function parameters are derived from

$$\tilde{d}_0 = \frac{ln(\hat{A}^{nlsur}) + \left(1 - \hat{\beta}^{nlsur}\right) ln\left(\hat{\alpha}^{nlsur}\right) + \hat{\beta}^{nlsur} ln\left(\hat{\beta}^{nlsur}\right)}{1 - \left(\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}\right)}, \quad \tilde{c}_0 = \tilde{d}_0 + ln\left(\frac{\tilde{d}_2}{1 + \tilde{d}_1}\right)$$

$$\tilde{d}_1 = \frac{-\left(1 - \hat{\beta}^{nlsur}\right)}{1 - \left(\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}\right)} < 0, \qquad \tilde{c}_1 = 1 + \tilde{d}_1 < 0$$

$$\tilde{d}_2 = \frac{-\hat{\beta}^{nlsur}}{1 - \left(\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}\right)} < 0, \qquad \tilde{c}_2 = \hat{d}_2 - 1 < 0$$

$$\tilde{d}_3 = \frac{\hat{g}^{nlsur}}{1 - \left(\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}\right)} > 0, \qquad \tilde{c}_3 = \tilde{d}_3 > 0.$$

In this case we would have to separately estimate the standard errors of the derived estimates of the long-run input demand function parameters by the delta method or bootstrapping.

Suppose that no data are available on K_t. As long as data are available on p_t, L_t, w_t, and r_t, one can estimate the input demand function for labor under long-run profit maximization and recover all of the input demand function parameters and production function parameters. Consider OLS direct estimation of the long-run input demand function for labor:

$$ln(L_t) = d_0 + d_1 ln\left(\frac{w_t}{p_t}\right) + d_2 ln\left(\frac{r_t}{p_t}\right) + d_3 t + u_{lt}.$$

As shown below, recovery of the parameters for the long-run input demand function for capital/nonlabor is straightforward:

$\tilde{c}_0 = \tilde{d}_0^{ols} + ln\left(\frac{\tilde{d}_2^{ols}}{1 + \tilde{d}_1^{ols}}\right)$
$\tilde{c}_1 = 1 + \tilde{d}_1^{ols} < 0$
$\tilde{c}_2 = \hat{d}_2^{ols} - 1$
$\tilde{c}_3 = \tilde{d}_3^{ols}.$

The estimated standard errors for the estimators $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ are identical to those of the parameter estimators $\tilde{d}_1^{ols}, \hat{d}_2^{ols}, \hat{d}_3^{ols}$ because these sets of estimators are linearly related to one another. However, the estimated standard error on the constant term $\tilde{c}_0$ is a bit more complicated because of the nonlinear relationship with the estimator $\hat{d}_1^{ols}$. Therefore, the standard error for $\tilde{c}_0$ has to be estimated by the delta method or possibly by bootstrapping.

If one preferred to estimate the production function parameters directly from the long-run demand function for labor under profit maximization, one would use NLS to estimate:

$$ln(L_t) = \left(\frac{-1}{1-\alpha-\beta}\right)\left\{\left[-ln(\alpha A) - \beta ln\left(\frac{\beta}{\alpha}\right)\right] + (1-\beta)\, ln\left(\frac{w_t}{p_t}\right) + \beta ln\left(\frac{r_t}{p_t}\right) - gt\right\} + u_{lt}.$$

Of course one could estimate the parameters of the long-run input demand functions indirectly from the NLS estimates of the production function parameters.

$\tilde{d}_0 = \dfrac{ln(\hat{A}^{nls}) + \left(1 - \hat{\beta}^{nls}\right) ln\left(\hat{\alpha}^{nls}\right) + \hat{\beta}^{nls} ln\left(\hat{\beta}^{nls}\right)}{1 - \left(\hat{\alpha}^{nls} + \hat{\beta}^{nls}\right)}$	$\tilde{c}_0 = \tilde{d}_0 + ln\left(\dfrac{\tilde{d}_2}{1 + \tilde{d}_1}\right)$
$\tilde{d}_1 = \dfrac{-\left(1 - \hat{\beta}^{nls}\right)}{1 - \left(\hat{\alpha}^{nls} + \hat{\beta}^{nls}\right)} < 0$	$\tilde{c}_1 = 1 + \tilde{d}_1 < 0$
$\tilde{d}_2 = \dfrac{-\hat{\beta}^{nls}}{1 - \left(\hat{\alpha}^{nls} + \hat{\beta}^{nls}\right)} < 0$	$\tilde{c}_2 = \hat{d}_2 - 1 < 0$
$\tilde{d}_3 = \dfrac{\hat{g}^{nlsur}}{1 - \left(\hat{\alpha}^{nlsur} + \hat{\beta}^{nlsur}\right)} > 0$	$\tilde{c}_3 = \tilde{d}_3 > 0.$

Another possible estimation scenario with exogenous output price and lack of data on K_t is one in which we can plausibly claim that K_t did not vary over the period of the data. In this case we could appeal to the short-run profit maximization model:

$$ln(L_t) = \eta_0 + \eta_1 ln\left(\frac{w_t}{p_t}\right) + \eta_2 t + u_{lt}.$$

As long as u_{lt} is i.i.d., the short-run demand for labor function can be estimated by OLS. The correspondence between the short-run labor demand function parameters and the production function parameters are described below:

$$\eta_0 = \left(\frac{1}{1-\alpha}\right)[ln(\alpha A) + \beta ln(K_0)]$$
$$\eta_1 = \frac{-1}{1-\alpha} < 0$$
$$\eta_2 = \frac{g}{1-\alpha} > 0,$$

where K_0 is some fixed level of the nonlabor inputs.

Perhaps not surprisingly, only the production function parameters α and g can be identified in this model. OLS estimation of the short-run labor demand function can be used to obtain estimates of α and g:

$$\tilde{\alpha} = \frac{1+\hat{\eta}_1^{ols}}{\hat{\eta}_1^{ols}}$$
$$\tilde{g} = \frac{-\hat{\eta}_2^{ols}}{\hat{\eta}_1^{ols}} > 0.$$

7 Exogenous Budget and Input Demands For a Public Agency

The empirical input demand functions for a public agency are very simple with a CD technology:

$$ln\left(\frac{w_t L_t}{C_t}\right) = \phi_1 + u_{lt}$$
$$ln\left(\frac{r_t K_t}{C_t}\right) = \phi_2 + u_{kt},$$

where $\phi_1 = ln\left(\frac{\alpha}{\alpha+\beta}\right)$ and $\phi_2 = ln\left(\frac{\beta}{\alpha+\beta}\right)$. Clearly, none of the production function parameters are identified in this model. The data requirements on the other hand are minimal since the budget share of only one input is needed. Assuming that labor's share of the agency's budget is known, we can easily infer the nonlabor share of the budget: $\frac{r_t K_t}{C_t} = 1 - \frac{w_t L_t}{C_t}$.

All that can be identified is the parameter ratio $\frac{\beta}{\alpha}$ which can be efficiently estimated by NLSUR. To see this note that

$$\phi_1 = \ln\left(\frac{\alpha}{\alpha+\beta}\right)$$
$$= \ln\left(\frac{1}{1+e^{\gamma}}\right),$$

where $e^{\gamma} = \left(\dfrac{\beta}{\alpha}\right)$ so that $\gamma = ln\left(\dfrac{\beta}{\alpha}\right)$. It follows that

$$\begin{aligned}\phi_2 &= ln\left(\frac{\beta}{\alpha+\beta}\right)\\ &= \ln\left(\frac{1}{1+e^{-\gamma}}\right).\end{aligned}$$

The two equation model could be estimated by NLSUR is given by

$$\begin{aligned}ln\left(\frac{w_t L_t}{C_t}\right) &= \ln\left(\frac{1}{1+e^{\gamma}}\right) + u_{lt}\\ ln\left(\frac{r_t K_t}{C_t}\right) &= \ln\left(\frac{1}{1+e^{-\gamma}}\right) + u_{kt}.\end{aligned}$$

Notice that

$$\begin{aligned}ln\left(\frac{r_t K_t}{C_t}\right) - ln\left(\frac{w_t L_t}{C_t}\right) &= ln\left(\frac{K_t}{L_t}\right) - ln\left(\frac{w_t}{r_t}\right),\\ \phi_2 - \phi_1 &= ln\left(\frac{\beta}{\alpha+\beta}\right) - ln\left(\frac{\alpha}{\alpha+\beta}\right)\\ &= ln\left(\frac{\beta}{\alpha}\right) = \gamma.\end{aligned}$$

This formulation reduces back to the elasticity of substitution specification of the relative input demand function:

$$\begin{aligned}ln\left(\frac{r_t K_t}{C_t}\right) - ln\left(\frac{w_t L_t}{C_t}\right) &= \phi_2 - \phi_1 + u_{lt} - u_{kt}\\ &= \gamma + u_{klt}.\end{aligned}$$

8 Endogenous Output and Output Prices

In this section we examine how the demand for labor is affected by competitive market conditions. Again we return to a simple, stylized market demand specification:

$$ln(Q_t^d) = \theta_0 + \theta_1 ln\left(\frac{p_t}{y_t}\right) + \theta_2 t + \varepsilon_{dt},$$

where $\theta_1 < 0$ and y_t is some measure of consumer income. A straightforward way to obtain the empirical market supply equation is to substitute the stochastic long-run input demand functions for L_t and K_t in the stochastic CD production function, collect terms, and take logs. The resulting market supply function may be expressed by

$$ln\left(Q_t^s\right) = \lambda_0 + \lambda_1 ln\left(\frac{w_t}{p_t}\right) + \lambda_2 ln\left(\frac{r_t}{p_t}\right) + \lambda_3 t + \varepsilon_{st},$$

where $\varepsilon_{st} = \alpha u_{lt} + \beta u_{kt} + u_{st}$,

$$\lambda_1 = \frac{-\alpha}{1-(\alpha+\beta)} < 0$$

$$\lambda_2 = \frac{-\beta}{1-(\alpha+\beta)} < 0$$

$$\lambda_3 = \frac{g}{1-(\alpha+\beta)} > 0$$

$$\lambda_0 = \left[\frac{\alpha+\beta}{1-(\alpha+\beta)}\right] ln\left(\frac{\alpha+\beta}{\psi}\right)$$

$$\psi = \left[A^{\frac{-1}{\alpha+\beta}}\right]\left[\left(\frac{\beta}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right].$$

Since price and quantity are jointly determined in the market, we can solve for the reduced form equations for price and quantity by equating market demand and supply;

$$ln(Q_t^d) = ln(Q_t^s)$$

$\Rightarrow$

$$ln(p_t) = \pi_{0p} + \pi_{1p} ln\left(w_t\right) + \pi_{2p} ln\left(r_t\right) + \pi_{3p} t + \pi_{4p} ln(y_t) + v_{pt},$$

where

$$\pi_{0p} = \frac{\lambda_0 - \theta_0}{\theta_1 + \lambda_1 + \lambda_2}$$

$$\pi_{1p} = \frac{\lambda_1}{\theta_1 + \lambda_1 + \lambda_2} > 0$$

$$\pi_{2p} = \frac{\lambda_2}{\theta_1 + \lambda_1 + \lambda_2} > 0$$

$$\pi_{3p} = \frac{\lambda_3 - \theta_2}{\theta_1 + \lambda_1 + \lambda_2} \gtreqless 0$$

$$\pi_{4p} = \frac{\theta_1}{\theta_1 + \lambda_1 + \lambda_2} > 0$$

$$v_{pt} = \frac{\varepsilon_{st} - \varepsilon_{dt}}{\theta_1 + \lambda_1 + \lambda_2}.$$

Note the following parameter restriction for the reduced form price equation: $\pi_{1p} + \pi_{2p} + \pi_{4p} = 1 \Rightarrow \pi_{4p} = 1 - \pi_{1p} - \pi_{2p}$. The reduced form price equation incorporating this restriction is given by

$$ln\left(\frac{p_t}{y_t}\right) = \pi_{0p} + \pi_{1p} ln\left(\frac{w_t}{y_t}\right) + \pi_{2p} ln\left(\frac{r_t}{y_t}\right) + \pi_{3p} t + v_{pt}$$

Upon substitution for $ln(p_t)$ in the supply equation and collecting terms, we obtain the reduced form output equation:

$$ln\left(Q_t\right) = \pi_{0q} + \pi_{1q} ln\left(\frac{w_t}{y_t}\right) + \pi_{2q} ln\left(\frac{r_t}{y_t}\right) + \pi_{3q} t + v_{qt}$$

where

$$\pi_{0q} = \theta_0 + \theta_1 \pi_{0p}$$

$$\pi_{1q} = \theta_1 \pi_{1p} < 0$$

$$\pi_{2q} = \theta_1 \pi_{2p} < 0$$

$$\pi_{3q} = \theta_2 + \theta_1 \pi_{3p} \gtreqless 0$$

$$\upsilon_{qt} = \upsilon_{pt} + \varepsilon_{dt}.$$

There are a variety of estimation strategies for recovering labor and nonlabor input demand function parameters. The structural output demand and supply parameters are identified from

$$\pi_{4p} = 1 - \pi_{1p} - \pi_{2p} = \frac{\theta_1}{\theta_1 + \lambda_1 + \lambda_2}$$
$$\lambda_1 = \frac{\pi_{1q}}{\pi_{4p}}$$
$$\lambda_2 = \frac{\pi_{2q}}{\pi_{4p}}$$
$$\theta_1 = \frac{\lambda_1}{\pi_{1p}} - \lambda_1 - \lambda_2 = \frac{\lambda_2}{\pi_{2p}} - \lambda_1 - \lambda_2$$
$$\theta_2 = \pi_{3q} - \theta_1 \pi_{3p}$$
$$\theta_0 = \pi_{0q} - \theta_1 \pi_{0p}$$
$$\lambda_0 = (\pi_{0p})(\theta_1 + \lambda_1 + \lambda_2) + \theta_0.$$

If data on L_t and K_t were not available one could still estimate the reduced form equations by OLS or more efficiently by NLSUR estimation with cross-equation restrictions:

$$ln\left(\frac{p_t}{y_t}\right) = \pi_{0p} + \pi_{1p} ln\left(\frac{w_t}{y_t}\right) + \pi_{2p} ln\left(\frac{r_t}{y_t}\right) + \pi_{3p} t + \upsilon_{pt}$$
$$ln\left(Q_t\right) = \theta_0 + \theta_1 \left[\pi_{0p} + \pi_{1p} ln\left(\frac{w_t}{y_t}\right) + \pi_{2p} ln\left(\frac{r_t}{y_t}\right) + \theta_1 \pi_{3p} t\right] + \theta_2 t + \upsilon_{qt}.$$

The output supply parameters would then be estimated according to

$$\tilde{\lambda}_0 = \hat{\theta}_0 + \frac{\hat{\theta}_1 \hat{\pi}_{0p}}{1 - \hat{\pi}_{1p} - \hat{\pi}_{2p}}$$
$$\tilde{\lambda}_1 = \frac{\hat{\theta}_1 \hat{\pi}_{1p}}{1 - \hat{\pi}_{1p} - \hat{\pi}_{2p}}$$
$$\tilde{\lambda}_2 = \frac{\hat{\theta}_1 \hat{\pi}_{2p}}{1 - \hat{\pi}_{1p} - \hat{\pi}_{2p}}$$
$$\tilde{\lambda}_3 = \hat{\theta}_2 + \frac{\hat{\theta}_1 \hat{\pi}_{3p}}{1 - \hat{\pi}_{1p} - \hat{\pi}_{2p}}.$$

It should come as no surprise that the production function parameters can be recovered from the estimated output supply function parameters since the latter are derived from the CD technology:

$$\tilde{\alpha} = \frac{-\tilde{\lambda}_1}{1 - \left(\tilde{\lambda}_1 + \tilde{\lambda}_2\right)}$$

$$\tilde{\beta} = \frac{-\tilde{\lambda}_2}{1 - \left(\tilde{\lambda}_1 + \tilde{\lambda}_2\right)}$$

$$\tilde{g} = \frac{\tilde{\lambda}_3}{1 - \left(\tilde{\lambda}_1 + \tilde{\lambda}_2\right)}$$

$$\tilde{A} = exp\left\{\left[1 - \left(\tilde{\alpha} + \tilde{\beta}\right)\right] \tilde{\lambda}_0 - \left(\tilde{\alpha} + \tilde{\beta}\right) ln \left(\tilde{\alpha} + \tilde{\beta}\right) + \left(\tilde{\alpha} + \tilde{\beta}\right) ln \left[\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{\frac{-\tilde{\beta}}{\tilde{\alpha} + \tilde{\beta}}} + \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{\frac{\tilde{\alpha}}{\tilde{\alpha} + \tilde{\beta}}}\right]\right\}$$

Labor economists would be most interested in recovery of the input demand function parameters, especially the labor demand parameters:

$$\tilde{d}_0 = \frac{ln\left(\tilde{A}\right) + \left(1-\tilde{\beta}\right) ln\left(\tilde{\alpha}\right) + \tilde{\beta} ln\left(\tilde{\beta}\right)}{1-\left(\tilde{\alpha}+\tilde{\beta}\right)}, \quad \tilde{c}_0 = \tilde{d}_0 + ln\left(\frac{\tilde{d}_2}{1+\tilde{d}_1}\right)$$

$$\tilde{d}_1 = \frac{-\left(1-\tilde{\beta}\right)}{1-\left(\tilde{\alpha}+\tilde{\beta}\right)} < 0, \qquad \tilde{c}_1 = 1 + \tilde{d}_1 > 0$$

$$\tilde{d}_2 = \frac{-\tilde{\beta}}{1-\left(\tilde{\alpha}+\tilde{\beta}\right)} < 0, \qquad \tilde{c}_2 = \hat{d}_2 - 1 < 0$$

$$\tilde{d}_3 = \frac{\tilde{g}}{1-\left(\tilde{\alpha}+\tilde{\beta}\right)} > 0, \qquad \tilde{c}_3 = \tilde{d}_3 > 0.$$

Suppose we do not have data on Q_t. Another estimation strategy is to estimate the long-run input demand functions (9) and (10) using IV or 3SLS with cross-equation restrictions. The endogenous variables in the model are $ln(L_t), ln(K_t), ln(p_t)$,and $ln(Q_t)$, while the exogenous variables are $t, ln(w_t), ln(r_t)$,and $ln(y_t)$. If we were also lacking data on K_t, we could just estimate the demand function for labor using IV.

9 Long-run Labor Demand Effects of Wages and Technology

In an industry setting, changes in the wages faced by all firms would have direct (substitution) and indirect (scale) effects on labor demand. Consider the effect of an exogenous change in the wage. The final impact on labor demand will comprise a direct effect and an indirect effect derived from the wage effect on market output price.

The exogenous wage effect on the market price is determined according to

$$\begin{aligned}
\Delta ln(p) &= \pi_{1p}\Delta ln(w) \\
&= \left(\frac{\lambda_1}{\theta_1 + \lambda_1 + \lambda_2}\right)\Delta ln(w) \\
&= \left[\frac{-\alpha}{1-(\alpha+\beta)}\right]\left[\frac{1}{\theta_1 - \frac{(\alpha+\beta)}{1-(\alpha+\beta)}}\right]\Delta ln(w) \\
&= \frac{-\alpha}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)}\Delta ln(w),
\end{aligned}$$

where $\frac{-\alpha}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)} > 0$. It is instructive to consider three special cases. The first case is one in which output market demand is perfectly inelastic, $\theta_1 = 0$. It is readily seen that the price change is determined by

$$\Delta ln(p) = \frac{\alpha}{(\alpha+\beta)}\Delta ln(w).$$

The second case is one in which the consumer elasticity of demand in the market is unitary elastic, $\theta_1 = -1$. The price change in this case is determined according to

$$\Delta ln(p) = \alpha\Delta ln(w).$$

The third special case is one in which market demand is infinitely elastic, $\theta_1 = -\infty$. Here, there is no wage impact on output price, only on market quantity:

$$\Delta ln(p) = 0.$$

Next consider the total effect of a change in w on labor demand:

$$\begin{aligned}
\Delta ln(L) &= \left[\frac{-1}{1-(\alpha+\beta)}\right][(1-\beta)\,\Delta ln(w) - \Delta ln(p)] \\
&= \left[\frac{-1}{1-(\alpha+\beta)}\right]\left[(1-\beta) + \frac{\alpha}{\theta_1\,[1-(\alpha+\beta)]-(\alpha+\beta)}\right]\Delta ln(w).
\end{aligned}$$

Again, we consider the three special cases for the consumer elasticity of demand for output. In the case of perfectly inelastic demand, $\theta_1 = 0$, there is no quantity adjustment. All of the wage change effect operates through the substitution effect between labor and the nonlabor inputs. To see this note that

$$\begin{aligned}
\Delta ln(L) &= \left[\frac{-1}{1-(\alpha+\beta)}\right][(1-\beta)\,\Delta ln(w) - \Delta ln(p)] \\
&= \left[\frac{-1}{1-(\alpha+\beta)}\right]\left[(1-\beta)\,\Delta ln(w) - \frac{\alpha}{(\alpha+\beta)}\Delta ln(w)\right] \\
&= \frac{-\beta}{\alpha+\beta}\Delta ln(w).
\end{aligned}$$

For the special case of unitary elasticity of consumer demand, $\theta_1 = -1$, the final wage impact on labor demand is also unitary elastic. That is, the resulting percentage change in labor demand is equal but of opposite sign to the exogenous wage change:

$$\begin{aligned}
\Delta ln(L) &= \left[\frac{-1}{1-(\alpha+\beta)}\right][(1-\beta)\,\Delta ln(w) - \alpha\Delta ln(w)] \\
&= -\Delta ln(w).
\end{aligned}$$

When the consumer elasticity of demand is infinite, $\theta_1 = -\infty$, output price does not change. In this scenario there is no price adjustment in response to changed wages so that the effect on labor demand is given by

$$\Delta ln(L) = \left[\frac{-1}{1-(\alpha+\beta)}\right]\left[(1-\beta)\,\Delta ln(w) - \Delta ln(p)\right]$$
$$\left[\frac{-(1-\beta)}{1-(\alpha+\beta)}\right]\Delta ln(w).$$

Because labor demand effects are smaller when output price can adjust to exogenous wage changes,

$$\left|\left[\frac{-1}{1-(\alpha+\beta)}\right]\left[(1-\beta)+\frac{\alpha}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)}\right]\right| < \left|\left[\frac{-(1-\beta)}{1-(\alpha+\beta)}\right]\right|,$$

an unintended consequence of imposing price controls to prevent prices from rising in response to an increase in the wage is that the reduction in employment is even larger.

Now consider the effects of neutral technological change on prices and labor demand. In this case we let $\Delta t = 1$. The effect on price is seen to be

$$\begin{aligned}\Delta ln(p) &= \pi_{3p}\\ &= \frac{\lambda_3-\theta_2}{\theta_1+\lambda_1+\lambda_2}\\ &= \frac{g-\theta_2\left[1-(\alpha+\beta)\right]}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)} \gtreqless 0.\end{aligned}$$

This indeterminancy of the passage of time arises because there could be exogenous trends in consumer preferences for the product that are either increasing or decreasing over time. As a simplification consider the case in which there is no pure consumer trend in demand, $\theta_2 = 0$. The trended price effect arises purely from neutral technological change:

$$\Delta ln(p) = \frac{g}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)} < 0.$$

Thus, neutral technological change would lead to falling prices over time. The final impact on labor demand will again comprise a direct effect and an indirect effect derived from the effect of neutral technological change on market output price.

It is instructive to consider the implications for price changes of the three special cases for market demand elasticity. When market demand is perfectly inelastic, $\theta_1 = 0$, the impact on output price is seen to be

$$\Delta ln(p) = \frac{-g}{\alpha + \beta} < 0.$$

In the case of unitary consumer elasticity of demand, $\theta_1 = -1$, output price will fall by the rate of neutral technological change:

$$\Delta ln(p) = -g < 0.$$

If the market elasticity of demand is infinitely inelastic, $\theta_1 = -\infty$, there is of course no change in price:

$$\Delta ln(p) = 0.$$

Now we are in a position to determine the total impact of neutral technological change on labor demand:

$$\begin{aligned}\Delta ln(L) &= \left[\frac{1}{1-(\alpha+\beta)}\right][\Delta ln(p) + g] \\ &= \left[\frac{g}{1-(\alpha+\beta)}\right]\left[\frac{1}{\theta_1\left[1-(\alpha+\beta)\right]-(\alpha+\beta)} + 1\right] \gtreqless 0.\end{aligned}$$

In general the labor demand effects of technological change in the industry is indeterminate and depends critically on the market elasticity of demand. It is instructive to consider the three special cases of market demand elasticity. When market demand is perfectly inelastic,

$\theta_1 = 0$, neutral technological change increases labor demand:

$$\begin{aligned}\Delta ln(L) &= \left[\frac{1}{1-(\alpha+\beta)}\right][\Delta ln(p) + g] \\ &= \left[\frac{1}{1-(\alpha+\beta)}\right]\left[\frac{-g}{\alpha+\beta} + g\right] \\ &= \frac{-g}{\alpha+\beta} < 0.\end{aligned}$$

When market demand is unitary elastic, $\theta_1 = -1$, there is no effect of neutral technological change on labor demand:

$$\begin{aligned}\Delta ln(L) &= \left[\frac{1}{1-(\alpha+\beta)}\right][\Delta ln(p) + g] \\ &= \left[\frac{1}{1-(\alpha+\beta)}\right][-g + g] \\ &= 0.\end{aligned}$$

In the case of infinitely elastic market demand, $\theta_1 = -\infty$, neutral technological change reduces labor demand:

$$\begin{aligned}\Delta ln(L) &= \left[\frac{1}{1-(\alpha+\beta)}\right][\Delta ln(p) + g] \\ &= \left[\frac{1}{1-(\alpha+\beta)}\right][0 + g] \\ &= \frac{g}{1-(\alpha+\beta)} > 0.\end{aligned}$$

More generally it can be shown that if consumer demand is relatively elastic, $-\theta_1 > 1$, employment will actually rise ($\Delta ln(L) > 0$) with neutral technological change. This is because output demanded increases by a larger percentage than the price reduction associated with neutral technological change. On the other hand, relatively inelastic market demand, $-\theta_1 < 1$, is associated with reductions in labor demand ($\Delta ln(L) < 0$) from neutral technological change. In this case output demanded rises proportionately less than the reduction in price associated with neutral technological change. A knife-edge case is one in which consumer demand is unitary elastic, $-\theta_1 = 1$. In this scenario, there is no change in labor demand ($\Delta ln(L) = 0$) because the percentage rise in output demanded just offsets the reduction in price associated with neutral technological change.