

The Econometrics of Decompositions

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I. BASIC DECOMPOSITIONS

Background reading:

Blinder, Alan S. (1973), "Wage Discrimination: Reduced Form and Structural Estimates," *Journal of Human Resources* 8: 436-455.

Oaxaca, Ronald L. (1973), "Male-Female Wage Differentials in Urban Labor Markets," *International Economic Review*. 14(3): 693-709.

Greene, William H. (2003), *Econometric Analysis*, Fifth Edition (International Edition), Chapter 3.

For purposes of illustration only, the decomposition techniques and issues are applied to the case of decomposing (log) wage differentials between men and women into explained and unexplained differences. However, it is clear that the approaches described below apply to any attempt to decompose mean sample differences between any two categories of observations, e.g. union workers versus nonunion workers, manufacturing firms versus non manufacturing firms, public sector workers versus private sector workers, workers in Mumbles versus workers in Cardiff.

From the properties of *OLS* we have $\bar{Y} = \bar{X}'\hat{\beta}$, where \bar{Y} is $T \times 1$, \bar{X}' is $1 \times k$, and $\hat{\beta}$ is $k \times 1$. A standard male/female wage decomposition is

$$\bar{Y}_m - \bar{Y}_f = (\bar{X}_m - \bar{X}_f)' \hat{\beta}_m + \bar{X}_f' (\hat{\beta}_m - \hat{\beta}_f). \quad (1)$$

This decomposition assumes that the m (male) structure is the norm. Accordingly, the term $(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m$ represents the "explained" differential. Therefore, the term $\bar{X}_f' (\hat{\beta}_m - \hat{\beta}_f)$ represents the "unexplained" differential. In some circles and in some contexts this term is interpreted as a measure of discrimination.

We might think of $\bar{Y}_f^0 = \bar{X}_f' \hat{\beta}_m$ as the mean competitive, nondiscriminatory (log) wage for females. In this case $\bar{Y}_m^0 = \bar{Y}_m = \bar{X}_m' \hat{\beta}_m$ since the male wage structure is the norm. The decomposition can be equivalently stated as

$$\begin{aligned} \bar{Y}_m - \bar{Y}_f &= (\bar{Y}_m^0 - \bar{Y}_f^0) + (\bar{Y}_f^0 - \bar{Y}_f) \\ &= (\bar{Y}_m - \bar{Y}_f^0) + (\bar{Y}_f^0 - \bar{Y}_f) \end{aligned}$$

Typically in wage regressions Y is the log of the wage, so that

$$\bar{Y} = \left(\sum_{t=1}^T \ln(w_t) \right) / T = \ln(\tilde{w}),$$

where \tilde{w} is the geometric mean. In this situation

$$\begin{aligned} \bar{Y}_m - \bar{Y}_f &= \ln(\tilde{w}_m) - \ln(\tilde{w}_f) \\ &= \ln(G + 1) \end{aligned}$$

where $G = \frac{\tilde{w}_m}{\tilde{w}_f} - 1$ is the gross (unadjusted) wage differential. Along these lines we can view the explained gap as the gap attributable to qualifications differences, i.e.

$$\begin{aligned} (\bar{X}_m - \bar{X}_f)' \hat{\beta}_m &= \bar{Y}_m - \bar{Y}_f^0 \\ &= \ln(\tilde{w}_m) - \ln(\tilde{w}_f^0) \\ &= \ln(Q_m + 1) \end{aligned}$$

where $Q_m = \frac{\tilde{w}_m}{\tilde{w}_f^0} - 1$ is the wage differential attributable to differences in qualifications when using the male wage structure as the norm. This leaves the unexplained differential, i.e.

$$\begin{aligned}\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) &= (\bar{Y}_f^0 - \bar{Y}_f) \\ &= \ln(\tilde{w}_f^0) - \ln(\tilde{w}_f) \\ &= \ln(D_m + 1)\end{aligned}$$

where $D_m = \frac{\tilde{w}_f^0}{\tilde{w}_f} - 1$ is the wage differential that is unexplained (discrimination ?) when using the male wage structure as the norm. With this notation in hand, we arrive at the following accounting identity:

$$\ln(G + 1) = \ln(Q_m + 1) + \ln(D_m + 1).$$

An alternative decomposition is given by

$$\bar{Y}_m - \bar{Y}_f = (\bar{X}_m - \bar{X}_f)' \hat{\beta}_f + \bar{X}'_m (\hat{\beta}_m - \hat{\beta}_f)$$

This decomposition assumes that the f structure is the norm. Now the term $(\bar{X}_m - \bar{X}_f)' \hat{\beta}_f$ measures the "explained" differential and $\bar{X}'_m (\hat{\beta}_m - \hat{\beta}_f)$ measures the "unexplained" differential. Here we can think of $\bar{Y}_m^0 = \bar{X}'_m \hat{\beta}_f$ as the mean competitive, nondiscriminatory (log) wage for males. In this case $\bar{Y}_f^0 = \bar{Y}_f = \bar{X}'_f \hat{\beta}_f$ since the female wage structure is the norm. Equivalently

$$\begin{aligned}\bar{Y}_m - \bar{Y}_f &= (\bar{Y}_m^0 - \bar{Y}_f^0) + (\bar{Y}_m - \bar{Y}_m^0) \\ &\quad (\bar{Y}_m^0 - \bar{Y}_f^0) + (\bar{Y}_m - \bar{Y}_m^0)\end{aligned}$$

The accounting identity that results from this decomposition is given by

$$\ln(G + 1) = \ln(Q_f + 1) + \ln(D_f + 1)$$

where

$$\begin{aligned}
 \ell n(Q_f + 1) &= \bar{Y}_m^0 - \bar{Y}_f \\
 &= (\bar{X}_m - \bar{X}_f)' \hat{\beta}_f \\
 &= \ell n(\tilde{w}_m^0) - \ell n(\tilde{w}_f).
 \end{aligned}$$

and

$$\begin{aligned}
 \ell n(D_f + 1) &= (\bar{Y}_m - \bar{Y}_m^0) \\
 &= \bar{X}_m' (\hat{\beta}_m - \hat{\beta}_f) \\
 &= \ell n(\tilde{w}_m) - \ell n(\tilde{w}_m^0).
 \end{aligned}$$

It is clear that $Q_f = \frac{\tilde{w}_m^0}{\tilde{w}_f} - 1$ is the wage differential attributable to differences in qualifications when using the female wage equation as the norm, and D_f is the wage differential that is unexplained.

The two sets of decompositions corresponding to the male wage structure as the norm or the female wage structure as the norm illustrate the index number problem with decompositions. In other words the separately calculated explained and unexplained components will in general differ depending on which structure is assumed to be the norm. Of course their sum will be the same.

II. IDENTIFICATION ISSUES

Background Reading:

Jones, F.L., (1983), "On Decomposing the Wage Gap: A Critical Comment on Blinder's Method," *Journal of Human Resources* 18, 126-130.

Oaxaca, Ronald L. and Michael Ransom (1999) "Identification in Detailed Wage Decompositions"(with Michael R. Ransom), *Review of Economics and Statistics*, vol. 81, no. 1, 154-157.

Greene, William H. (2003), *Econometric Analysis*, Fifth Edition (International Edition), Chapter 7 (pp.116-120).

Separately estimated (log) wage equations for males and females evaluated at the sample means are given by

$$\bar{Y}_m = \hat{\beta}_{mo} + \sum_{j=1}^N \bar{X}_m^{(j)'} \hat{\beta}_m^{(j)} ,$$
$$\bar{Y}_f = \hat{\beta}_{fo} + \sum_{j=1}^N \bar{X}_f^{(j)'} \hat{\beta}_f^{(j)} ,$$

where $\hat{\beta}_{io}$ is the estimated intercept term, $\hat{\beta}_i^{(j)}$ is a column vector of estimated slope coefficients for the set of regressors comprising the j th variable, and $\bar{X}_i^{(j)'}$ is a row vector of regressor means for the set of regressors comprising the j th variable. There are N variables defined by N sets of regressors, e.g. experience and experience squared would constitute the experience variable. If we adopt the m wage structure as the

norm, the gender wage gap is decomposed according to

$$\bar{Y}_m - \bar{Y}_f = \underbrace{\left(\hat{\beta}_{mo} - \hat{\beta}_{fo} \right) + \sum_{j=1}^N \bar{X}_f^{(j)'} \Delta \hat{\beta}^{(j)}}_{\text{discrimination}} + \underbrace{\sum_{j=1}^N \Delta \bar{X}^{(j)'} \hat{\beta}_m^{(j)}}_{\text{endowments}}, \quad (2)$$

where $\Delta \hat{\beta}^{(j)} = \hat{\beta}_m^{(j)} - \hat{\beta}_f^{(j)}$ and $\Delta \bar{X}^{(j)'} = \bar{X}_m^{(j)'} - \bar{X}_f^{(j)'}$. The contributions of the j th variable to discrimination and endowments are $\bar{X}_f^{(j)'} \Delta \hat{\beta}^{(j)}$ and $\Delta \bar{X}^{(j)'} \hat{\beta}_m^{(j)}$, and the contribution of the intercept term to the discrimination component is $(\hat{\beta}_{mo} - \hat{\beta}_{fo})$. Given the specification of the X 's, there is seemingly no ambiguity surrounding the decomposition. However, this is an illusion.

Consider the case in which a variable V defined by a set of dummy variables is added to the wage regressions, e.g. marital status. The set of dummy variable mean values are denoted by $\{\bar{V}_{ik} \mid k = 1, \dots, K_1\}$, where $\sum_{k=1}^{K_1} \bar{V}_{ik} = 1$, $i = m, f$. Without loss of generality the first dummy variable category (\bar{V}_{i1}) will initially serve as the left out reference group, e.g. married, spouse present. The separately estimated wage equations for men and women evaluated at the sample means can now be expressed as

$$\begin{aligned} \bar{Y}_m &= \hat{\beta}_{mo} + \sum_{k=2}^{K_1} \bar{V}_{mk} \hat{\delta}_{mk} + \sum_{j=1}^N \bar{X}_m^{(j)'} \hat{\beta}_m^{(j)} \\ &= \sum_{k=1}^{K_1} \bar{V}_{mk} \hat{\theta}_{mk} + \sum_{j=1}^N \bar{X}_m^{(j)'} \hat{\beta}_m^{(j)}, \end{aligned}$$

$$\begin{aligned} \bar{Y}_f &= \hat{\beta}_{fo} + \sum_{k=2}^{K_1} \bar{V}_{fk} \hat{\delta}_{fk} + \sum_{j=1}^N \bar{X}_f^{(j)'} \hat{\beta}_f^{(j)} \\ &= \sum_{k=1}^{K_1} \bar{V}_{fk} \hat{\theta}_{fk} + \sum_{j=1}^N \bar{X}_f^{(j)'} \hat{\beta}_f^{(j)}, \end{aligned}$$

where $\hat{\delta}_{ik} = \hat{\theta}_{ik} - \hat{\theta}_{i1}$, and our left out reference group choice implies the normalization

$\hat{\beta}_{io} = \hat{\theta}_{i1}$. Accordingly, the resulting wage decomposition is given by

$$\begin{aligned}
\bar{Y}_m - \bar{Y}_f &= \underbrace{\left(\hat{\beta}_{mo} - \hat{\beta}_{fo} \right) + \sum_{k=2}^{K_1} \bar{V}_{fk} \left(\hat{\delta}_{mk} - \hat{\delta}_{fk} \right) + \sum_{j=1}^N \bar{X}_f^{(j)'} \Delta \hat{\beta}^{(j)}}_{\text{discrimination}} \\
&\quad + \underbrace{\sum_{k=2}^{K_1} \left(\bar{V}_{mk} - \bar{V}_{fk} \right) \hat{\delta}_{mk} + \sum_{j=1}^N \Delta \bar{X}^{(j)'} \hat{\beta}_m^{(j)}}_{\text{endowments}} \\
&= \underbrace{\sum_{k=1}^{K_1} \bar{V}_{fk} \left(\hat{\theta}_{mk} - \hat{\theta}_{fk} \right) + \sum_{j=1}^N \bar{X}_f^{(j)'} \Delta \hat{\beta}^{(j)}}_{\text{discrimination}} \\
&\quad + \underbrace{\sum_{k=1}^{K_1} \left(\bar{V}_{mk} - \bar{V}_{fk} \right) \hat{\theta}_{mk} + \sum_{j=1}^N \Delta \bar{X}^{(j)'} \hat{\beta}_m^{(j)}}_{\text{endowments}} .
\end{aligned} \tag{3}$$

A number of things are immediately apparent from the decompositions described by (3). First, the estimated overall discrimination and the estimated overall endowment effect are invariant to the choice of left out reference group and to the suppression of the constant term in the absence of a left out reference group. That is, the alternative expressions for the estimated overall discrimination and endowment contributions in (3) are the same decompositions. Second, the contribution of the variable V to discrimination as estimated by $\sum_{k=2}^{K_1} \bar{V}_{fk} \left(\hat{\delta}_{mk} - \hat{\delta}_{fk} \right)$ is not invariant with respect to the choice of left out reference group. For example, designating the last dummy variable category (\bar{V}_{iK_1}) as the left out reference group (e.g. single, never married) would replace the immediately preceding measure of discrimination effects with $\sum_{k=1}^{K_1-1} \bar{V}_{fk} \left(\hat{\phi}_{mk} - \hat{\phi}_{fk} \right)$, where $\hat{\phi}_{ik} = \hat{\theta}_{ik} - \hat{\theta}_{iK_1}$. These two sets of estimates will not be the same because the intercept term is altered by the renormalization, $\hat{\beta}_{io} = \hat{\theta}_{iK_1}$. Third, it is possible to identify the contribution of V to discrimination as $\left(\hat{\beta}_{mo} - \hat{\beta}_{fo} \right) + \sum_{k=2}^{K_1} \bar{V}_{fk} \left(\hat{\delta}_{mk} - \hat{\delta}_{fk} \right) = \sum_{k=1}^{K_1} \bar{V}_{fk} \left(\hat{\theta}_{mk} - \hat{\theta}_{fk} \right)$, i.e. the intercept

contribution is part of the contribution of V to discrimination. This interpretation requires a normalizing restriction that holds that in the absence of variable V there would be no constant term. Last, the contribution of V to endowments is invariant with respect to the choice of left out reference group, i.e. $\sum_{k=2}^{K_1} (\bar{V}_{mk} - \bar{V}_{fk}) \hat{\delta}_{mk} = \sum_{k=1}^{K_1-1} (\bar{V}_{mk} - \bar{V}_{fk}) \hat{\phi}_{mk} = \sum_{k=1}^{K_1} (\bar{V}_{mk} - \bar{V}_{fk}) \hat{\theta}_{mk}$.

The analysis can be generalized to multiple sets of dummy variables. It is clear that if the constant term is suppressed in models with more than one set of dummy variables, all but one of the sets of dummy variables must have left out reference categories in order to avoid perfect multicollinearity. A number of implications follow from the generalization. First, alternative decompositions are equivalent in terms of the estimates of overall discrimination and the overall contribution of endowments. Therefore, the overall decomposition is invariant with respect to the choice of left out reference groups. Second, it can be shown that the *combined* estimated contributions of all sets of dummy variables to overall discrimination (inclusive of the constant term) and to overall endowment effects are invariant with respect to the choice of left out reference groups. Third, the separate contributions of sets of dummy variables to discrimination are not invariant with respect to the choice of left out reference groups. Fourth, unlike the case with only one set of dummy variables expressed by (3), there are no unique estimates of the separate contributions of the sets of dummy variables to overall discrimination.

Empirical Example

Our example is based on gender salary decompositions for a sample of full time, full-year U.S. college and university professors. The model consists of variables to measure total experience as a professor, seniority at current institution, and a set of dummy variables to indicate highest degree held, along with continuous measures of publication activity (journal articles, books, and collections), and dummy variables for a 12 month contract, teaching field, and race/ethnicity. We specify the “highest

Wage Decompositions with Alternative Reference Groups				
Variable	No Adv. Deg (Ref)		Ph.D. (Ref)	
	Disc	Endow	Disc	Endow
Constant	0.219	0.000	0.037	0.000
Seniority	-0.014	0.007	-0.014	0.007
Experience	0.064	0.074	0.064	0.074
Degree Type	-0.193	0.042	-0.011	0.042
Cont length	0.007	-0.005	0.007	-0.005
Pub Activity	-0.046	0.049	-0.046	0.049
Field	0.053	0.010	0.053	0.010
Race/Ethnic	-0.000	-0.001	-0.000	-0.001
Total	0.090	0.176	0.090	0.176

FIG. 1.

degree” variable with respect to two different reference groups. In one case, the reference group is those who have no advanced degree; the other reference group is those with a Ph.D. degree. The unadjusted differential is 26.6 log points. That is, women’s salaries are about 25 percent less than men’s. The decomposition table below illustrates the identification problem.

Differences in average qualifications between men and women explain 17.6 log points, so the estimate of discrimination is 9 log points. When No Advanced Degree is the reference group, the partial contribution of degree type to discrimination is -19.3 log points, and the contribution of constant term differences is 21.9 log points. For Ph.D. as the left out group, these contributions are -1.1 log points and 3.7 log points, respectively. The partial contribution of degree type to the endowment effect is 4.2 log points regardless of the left out reference group. This clearly demonstrates the identification problem—the choice of education category for the reference group is entirely arbitrary, yet the amount of discrimination that is attributed to degree type varies dramatically.

III. GENERALIZED WAGE DECOMPOSITIONS

Background Reading:

Cotton, Jeremiah (1988), "On the Decomposition of Wage Differentials," *Review of Economics and Statistics* 70: 236-243.

Neumark, David (1988), Employers' "Discriminatory Behavior and the Estimation of Wage Discrimination," *Journal of Human Resources* 23: 279-295.

Oaxaca, Ronald L. and Michael Ransom (1988), "Searching for the Effect of Unionism on the Wages of Union Workers and Nonunion Workers", *Journal of Labor Research*, IX (2): 139-148.

Oaxaca, Ronald L. and Michael Ransom (1994), "On Discrimination and the Decomposition of Wage Differentials," *Journal of Econometrics* 9: 5-21.

Reimers, Cordelia (1983), "Labor Market Discrimination against Hispanic and Black Men," *Review of Economics and Statistics* 65: 570-579.

The standard decomposition technique estimates only relative differences. In the case of discrimination estimates, we do not know how much of the unexplained (discriminatory) wage gap arises from favoritism toward one group of workers and how much arises from pure discrimination against the other group. If we let 'o' denote the absence of discrimination in a competitive labor market, the following relationships are implied by the log wage decompositions:

$$G = W_m/W_f - 1 \text{ (the unadjusted male/female wage differential)}$$

$$Q = W_m^o/W_f^o - 1 \text{ (the male/female wage differential attributable to qualifications)}$$

$$D = (W_m/W_f - W_m^o/W_f^o) / (W_m^o/W_f^o) \text{ represents the discrimination differential.}$$

In general we could write

$$\begin{aligned}
\ell n(G + 1) &= \ell n(D + 1) + \ell n(Q + 1) \\
&= \ell n(W_m/W_m^o) + \ell n(W_f^o/W_f) + \ell n(W_m^o/W_f^o) \\
&= \ell n(\delta_{mo} + 1) + \ell n(\delta_{of} + 1) + \ell n(Q + 1)
\end{aligned}$$

where $\delta_{mo} = W_m/W_m^o - 1$ (favoritism toward males) and $\delta_{of} = W_f^o/W_f - 1$ (pure discrimination against females).

In log terms the nondiscriminatory wages for men and women could be expressed as

$$\begin{aligned}
\ell n(\tilde{W}_m) &= \bar{X}'_m \hat{\beta}^* \\
\ell n(\tilde{W}_f) &= \bar{X}'_f \hat{\beta}^*
\end{aligned}$$

where $\hat{\beta}^*$ is the estimated parameter vector in the absence of discrimination. An operational wage decomposition for a sample of workers can be expressed as

$$\begin{aligned}
\ell n(G + 1) &= \ell n(\tilde{W}_m/\tilde{W}_m^o) + \ell n(\tilde{W}_f^o/\tilde{W}_f) + \ell n(\tilde{W}_m^o/\tilde{W}_f^o) \\
&= \bar{X}'_m (\hat{\beta}_m - \hat{\beta}^*) + \bar{X}'_f (\hat{\beta}^* - \hat{\beta}_f) + (\bar{X}'_m - \bar{X}'_f) \hat{\beta}^* \\
&= \ell n(\hat{\delta}_{mo} + 1) + \ell n(\hat{\delta}_{of} + 1) + \ell n(Q + 1).
\end{aligned}$$

We can quickly narrow down the possibilities for obtaining $\hat{\beta}^*$ to an infinite number. Fortunately, we can confine our attention to a smaller number of plausible possibilities. Assume that in the immediate aftermath of a sudden cessation of labor market discrimination, $\hat{\beta}^*$ would be a function of the currently estimated wage structures for males and females. A simple approximation would be to express $\hat{\beta}^*$ as a matrix weighted average of the vectors $\hat{\beta}_m$ and $\hat{\beta}_f$:

$$\hat{\beta}^* = \Omega \hat{\beta}_m + (I - \Omega) \hat{\beta}_f,$$

where Ω is an arbitrary $k \times k$ matrix and I is a $k \times k$ identity matrix. This still appears to admit an infinite number of possibilities. A reasonable choice for the weighting matrix is

$$\Omega = (X'X)^{-1}(X'_m X_m)$$

where $X'X = X'_m X_m + X'_f X_f$ is the cross product matrix for the combined sample of males and females. It is easily verified that $\hat{\beta}^*$ in this case is the *OLS* estimator applied to the combined sample:

$$\begin{aligned}\hat{\beta}^* &= (X'X)^{-1}X'Y \\ &= \hat{\beta}.\end{aligned}$$

There are some interesting special cases to consider:

$$\Omega = I \Rightarrow \hat{\beta}^* = \hat{\beta}_m, \hat{\delta}_{mo} = 0, \hat{D} = \hat{\delta}_{of} = \tilde{W}_f^o / \tilde{W}_f - 1.$$

$$\Omega = 0 \Rightarrow \hat{\beta}^* = \hat{\beta}_f, \hat{\delta}_{of} = 0, \hat{D} = \hat{\delta}_{mo} = \tilde{W}_m / \tilde{W}_m^o - 1.$$

Cotton (1988): $\Omega = \ell_m I$, where ℓ_m is the male proportion of the labor force.

Suppose the sample proportion of male workers is used, $\ell_m = T_m/T$.

Cotton's weighting scheme is equivalent to that of Oaxaca & Ransom in the following instance:

$$\begin{aligned}\Omega_c &= \Omega_o \Rightarrow (T_m/T)(X'X) = (X'_m X_m) \\ &\Rightarrow T^{-1}(X'X) = T_m^{-1}(X'_m X_m).\end{aligned}$$

This implies that the first and second sample moments for the regressors are identical for males and females. In particular this means that average characteristics are identical, so that all of the unadjusted differential is attributable to discrimination.

Reimers (1983): $\Omega = \frac{1}{2}I$. This is a special case of Cotton's weighting scheme,
i.e. $T_m/T = \frac{1}{2}$.

IV. DECOMPOSITION STANDARD ERRORS

Background reading:

Oaxaca, Ronald L. and Michael Ransom (1998) "Calculation of Approximate Variances for Wage Decomposition Differentials", *Journal of Economic and Social Measurement*, 24(1).

Greene, William H. (2003), *Econometric Analysis*, Fifth Edition (International Edition), Chapter 4 (pp. 53-54), Chapter 5 (p. 70), Appendix D (pp. 913-914).

We start with the variances of the log wage decomposition when adopting the male wage structure as the norm. Note

$$\begin{aligned} E[\ln(Q_m + 1)] &= (\bar{X}_m - \bar{X}_f)' E(\hat{\beta}_m) \\ &= (\bar{X}_m - \bar{X}_f)' \beta_m \end{aligned}$$

when we condition on \bar{X} . The true variance of $\ln(Q_m + 1)$ is given by

$$\begin{aligned} \text{var}[\ln(Q_m + 1)] &= E\left\{(\bar{X}_m - \bar{X}_f)' (\hat{\beta}_m - \beta_m) (\hat{\beta}_m - \beta_m)' (\bar{X}_m - \bar{X}_f)\right\} \\ &= (\bar{X}_m - \bar{X}_f)' E\left[(\hat{\beta}_m - \beta_m) (\hat{\beta}_m - \beta_m)'\right] (\bar{X}_m - \bar{X}_f) \\ &= (\bar{X}_m - \bar{X}_f)' \Sigma_{\hat{\beta}_m} (\bar{X}_m - \bar{X}_f) \end{aligned}$$

where $\Sigma_{\hat{\beta}_m} = \text{var}(\hat{\beta}_m)$ is the $k \times k$ variance/covariance matrix of $\hat{\beta}_m$. Next we seek an expression for the variance of $\ln(D_m + 1)$. Note

$$\begin{aligned} E[\ln(D_m + 1)] &= \bar{X}_f' [E(\hat{\beta}_m) - E(\hat{\beta}_f)] \\ &= \bar{X}_f' (\beta_m - \beta_f). \end{aligned}$$

Accordingly, the true variance of $\ell n(D_m + 1)$ is given by

$$\begin{aligned} \text{var}[\ell n(D_m + 1)] &= E \left\{ \left[\bar{X}'_f \left((\hat{\beta}_m - \beta_m) - (\hat{\beta}_f - \beta_f) \right) \right] \right. \\ &\quad \left. \cdot \left[\left((\hat{\beta}_m - \beta_m) - (\hat{\beta}_f - \beta_f) \right)' \bar{X}_f \right] \right\} \\ &= \bar{X}'_f \left(\Sigma_{\hat{\beta}_m} + \Sigma_{\hat{\beta}_f} \right) \bar{X}_f \end{aligned}$$

where $\Sigma_{\hat{\beta}_f} = \text{var}(\hat{\beta}_f)$ is the $k \times k$ variance/covariance matrix of $\hat{\beta}_f$. Note that $\text{cov}(\hat{\beta}_m, \hat{\beta}_f) = 0$. It is straight forward to show

$$\text{var}[\ell n(G + 1)] = \bar{X}'_m \Sigma_{\hat{\beta}_m} \bar{X}_m + \bar{X}'_f \Sigma_{\hat{\beta}_f} \bar{X}_f.$$

The true standard errors of $\ell n(G + 1)$, $\ell n(Q_m + 1)$, and $\ell n(D_m + 1)$ are simply the square roots of $\text{var}[\ell n(G + 1)]$, $\text{var}[\ell n(Q_m + 1)]$, and $\text{var}[\ell n(D_m + 1)]$. In practice the variance and standard errors are estimated by using the estimated values of $\Sigma_{\hat{\beta}_m}$ and $\Sigma_{\hat{\beta}_f}$.

We next consider the variances and standard errors for the decomposition terms corresponding to the decomposition that assumes that the f structure is the competitive, nondiscriminatory norm. In a parallel fashion to the above expressions, we obtain

$$\begin{aligned} E[\ell n(Q_f + 1)] &= (\bar{X}_m - \bar{X}_f)' \beta_f \\ E[\ell n(D_f + 1)] &= \bar{X}'_m (\beta_m - \beta_f) \end{aligned}$$

$$\begin{aligned} \text{var}[\ell n(Q_f + 1)] &= (\bar{X}_m - \bar{X}_f)' \Sigma_{\hat{\beta}_f} (\bar{X}_m - \bar{X}_f) \\ \text{var}[\ell n(D_f + 1)] &= \bar{X}'_m \left(\Sigma_{\hat{\beta}_m} + \Sigma_{\hat{\beta}_f} \right) \bar{X}_m. \end{aligned}$$

The true standard errors of $\ell n(Q_f + 1)$ and $\ell n(D_f + 1)$ are simply the square roots of $\text{var}[\ell n(Q_f + 1)]$ and $\text{var}[\ell n(D_f + 1)]$. In practice the variance and standard errors are estimated by using the estimated values of $\Sigma_{\hat{\beta}_m}$ and $\Sigma_{\hat{\beta}_f}$.

Suppose we are interested in placing standard errors on the differentials Q and D . As an example, consider the simple decomposition in which the m structure is the competitive, nondiscriminatory norm. We know that

$$\ell n(Q_m + 1) = (\bar{X}_m - \bar{X}_f)' \hat{\beta}_m$$

implies

$$Q_m = e^{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m} - 1$$

which is a nonlinear function of $\hat{\beta}_m$. In a simple case like this, one can apply the delta method which is a first order Taylor series approximation. In general $g(\hat{\theta}) \approx g(\theta) + g'(\theta) (\hat{\theta} - \theta)$, where $g'(\theta)$ is the first derivative of $g(\theta)$ with respect to θ . An asymptotic approximation yields

$$\begin{aligned} E[g(\hat{\theta})] &= g(\theta) \\ \text{var}[g(\hat{\theta})] &= [g'(\theta)] \text{var}(\hat{\theta}) [g'(\theta)]'. \end{aligned}$$

Under fairly general conditions

$$g(\hat{\theta}) \xrightarrow{a} N(g(\theta), [g'(\theta)] \text{var}(\hat{\theta}) [g'(\theta)]').$$

In practice θ is replaced by $\hat{\theta}$ for purposes of calculation. In our example

$$\begin{aligned} g(\hat{\theta}) &= Q_m \\ \hat{\theta} &= \hat{\beta}_m \\ \text{var}(\hat{\theta}) &= \text{var}(\hat{\beta}_m) = \Sigma_{\hat{\beta}_m} \\ g'(\hat{\theta}) &= \frac{\partial Q_m}{\partial \hat{\beta}_m} = e^{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m} (\bar{X}_m - \bar{X}_f)' = (Q_m + 1) (\bar{X}_m - \bar{X}_f)'. \text{ Accord-} \\ &\text{ingly,} \\ \text{var}(Q_m) &= (Q_m + 1)^2 (\bar{X}_m - \bar{X}_f)' \Sigma_{\hat{\beta}_m} (\bar{X}_m - \bar{X}_f). \end{aligned}$$

In the case of D_m we have

$$\ell n(D_m + 1) = \bar{X}'_f \left(\hat{\beta}_m - \hat{\beta}_f \right)$$

which implies

$$D_m = e^{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f)} + 1.$$

Application of the delta method yields

$$\text{var}(D_m) = (D_m + 1)^2 \bar{X}'_f \left(\Sigma_{\hat{\beta}_m} + \Sigma_{\hat{\beta}_f} \right) \bar{X}_f.$$

The construction of the variances for the decomposition differentials under the f structure is quite similar. Note that

$$\ell n(Q_f + 1) = (\bar{X}_m - \bar{X}_f)' \hat{\beta}_f$$

implies

$$Q_f = e^{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_f} - 1$$

and

$$\ell n(D_f + 1) = \bar{X}'_m \left(\hat{\beta}_m - \hat{\beta}_f \right)$$

implies

$$D_f = e^{\bar{X}'_m (\hat{\beta}_m - \hat{\beta}_f)} - 1.$$

It is straightforward to show

$$\text{var}(Q_f) = (Q_f + 1)^2 (\bar{X}_m - \bar{X}_f)' \Sigma_{\hat{\beta}_f} (\bar{X}_m - \bar{X}_f)$$

and

$$\text{var}(D_f) = (D_f + 1)^2 \bar{X}'_m \left(\Sigma_{\hat{\beta}_m} + \Sigma_{\hat{\beta}_f} \right) \bar{X}_m.$$

The estimated standard errors for these various differentials are obtained by taking the square roots of the variances and replacing the variance/covariance matrices for $\hat{\beta}_m$ and $\hat{\beta}_f$ with their estimated values. One can also obtain variances and standard errors for the favoritism and pure discrimination differentials for the generalized decomposition in an analogous fashion.

V. DECOMPOSITIONS WITH SELECTIVITY CORRECTIONS

Background reading:

Neuman, Shoshana and Ronald L. Oaxaca. (April 2001), "Gender Versus Ethnic Wage Differentials Among Professionals: Evidence from Israel," *Annales D'Économie et de Statistique*.

Greene, William H. (2003), *Econometric Analysis*, Fifth Edition (International Edition), Chapter 22 (pp. 780-787).

In this section we consider how decompositions are affected by corrections for sample selection. For example, working men and women may not be a random sample of the working age population. This sample selectivity may impart biases in wage equations unless the sample selection effects are taken into account when estimating the wage equation. The simplest approach is to first model the probability of employment as a probit. A two equation model arises for each gender group:

$$E_{ij}^* = Z'_{ij}\gamma_j + \varepsilon_{ij}$$

$$Y_{ij} = X'_{ij}\beta_j + u_{ij},$$

where for individual 'i' in the *j*th gender group, E_{ij}^* is a latent variable associated with employment, Z'_{ij} is a vector of the determinants of employment, Y_{ij} is the market wage (in logs), X'_{ij} is a vector of determinants of market wages, γ_j and β_j are the associated parameter vectors, and ε_{ji} and u_{ji} are *i.i.d* error terms that follow a bivariate normal distribution $(0, 0, \sigma_{\varepsilon_j}, \sigma_{u_j}, \rho_j)$. For identification purposes, the variance of ε_{ji} is normalized to 1.

While E_{ij}^* is unobserved as a continuous variable, market wages (Y_{ij}) are observed

when $E_{ij}^* > 0$. The probability of being employed is given by

$$\begin{aligned}\text{Pr ob}(E_{ij}^* > 0) &= \text{Pr ob}(\varepsilon_{ij} > -Z'_{ij}\gamma_j) \\ &= \Phi(Z'_{ij}\gamma_j),\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal C.D.F. (the variance of ε is normalized to 1). The market wage equation is estimated for $\{i \mid \varepsilon_{ij} > -Z'_{ij}\gamma_j\}$.

We have the familiar result that the expected wage of an employed worker is

$$\begin{aligned}E(Y_{ij} \mid E_{ij}^* > 0) &= X'_{ij}\beta_j + E(u_{ij} \mid \varepsilon_{ij} > -Z'_{ij}\gamma_j) \\ &= X'_{ij}\beta_j + \rho_j\sigma_{u_j}\lambda_{ij} \\ &= X'_{ij}\beta_j + \theta_j\lambda_{ij},\end{aligned}$$

where $\theta_j = \rho_j\sigma_{u_j}$, $\lambda_{ij} = \phi(Z'_{ij}\gamma_j)/\Phi(Z'_{ij}\gamma_j)$, and $\phi(\cdot)$ is the standard normal density function.

It is clear that correction for selectivity bias when comparing two demographic groups m and f requires a wage decomposition of the following sort (assuming the m wage structure is the norm):

$$\begin{aligned}\bar{Y}_m - \bar{Y}_f &= \left(\bar{X}'_m \hat{\beta}_m + \hat{\theta}_m \hat{\lambda}_m\right) - \left(\bar{X}'_f \hat{\beta}_f + \hat{\theta}_f \hat{\lambda}_f\right) \\ &= \bar{X}'_f \left(\hat{\beta}_m - \hat{\beta}_f\right) + \left(\bar{X}_m - \bar{X}_f\right)' \hat{\beta}_m \\ &\quad + \left(\hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f\right)\end{aligned}$$

The first two terms in the above decomposition are the familiar discrimination and endowment components, and the last term measures gender differences in the selection effects. A potentially critical issue is how to analyze and interpret this last term. One way to finesse the problem of what to do with the term $\left(\hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f\right)$ is to simply net out the estimated differences in conditional means from the overall wage

differential so that one is left with the familiar decomposition terms:

$$\begin{aligned} (\bar{Y}_m - \bar{Y}_f) - (\hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f) &= \bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) \\ &+ (\bar{X}_m - \bar{X}_f)' \hat{\beta}_m. \end{aligned}$$

One issue that arises is that this approach does not provide a decomposition of the *observed* wage differential $\bar{Y}_m - \bar{Y}_f$.

Of particular interest is the question of whether or not the term $(\hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f)$ should be subject to further decomposition into discrimination and endowment components, and if so, how should this be done? It is important to understand what gives rise to gender differences in the selection terms. Consider the following decomposition of the gender difference in the conditional mean error terms for the wage equations for the employed :

$$\begin{aligned} &\bar{E}(u_m | \varepsilon_m > -Z'_m \hat{\gamma}_m) - \bar{E}(u_f | \varepsilon_f > -Z'_f \hat{\gamma}_f) \\ &= \hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f \\ &= \hat{\theta}_m (\hat{\lambda}_f^0 - \hat{\lambda}_f) + \hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f^0) + (\hat{\theta}_m - \hat{\theta}_f) \hat{\lambda}_m, \end{aligned}$$

where $\hat{\lambda}_j = \sum_{i=1}^{N_j} \hat{\lambda}_{ij} / N_j$ and $\hat{\lambda}_{ij} = \phi(Z'_{ij} \hat{\gamma}_j) / \Phi(Z'_{ij} \hat{\gamma}_j)$ for $j = m, f$, $\hat{\lambda}_f^0 = \sum_{i=1}^{N_k} \hat{\lambda}_{if}^0 / N_f$, and $\hat{\lambda}_{if}^0 = \phi(Z'_{if} \hat{\gamma}_m) / \Phi(Z'_{if} \hat{\gamma}_m)$. The term $\hat{\lambda}_f^0$ is the mean value of the Inverse Mills Ratio (*IMR*) if females faced the same selection equation that the men face. The term $\hat{\theta}_m (\hat{\lambda}_f^0 - \hat{\lambda}_f)$ measures the effects of gender differences in the parameters of the probit selectivity equation on the male/female wage differential. The effects of gender differences in the variables that determine professional employment are measured by the term $\hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f^0)$. Finally, the effects of gender differences in the observed wage response to selection are captured by the term $(\hat{\theta}_m - \hat{\theta}_f) \hat{\lambda}_f$. Given that $\hat{\theta}_j = \hat{\rho}_j \hat{\sigma}_{u_j}$ and the parameters $\hat{\rho}_j$ and $\hat{\sigma}_{u_j}$ are identified, further decomposition of $\hat{\theta}_m - \hat{\theta}_f$ is

possible:

$$\hat{\theta}_m - \hat{\theta}_f = \hat{\rho}_m (\hat{\sigma}_{u_m} - \hat{\sigma}_{u_f}) + (\hat{\rho}_m - \hat{\rho}_f) \hat{\sigma}_{u_f} \quad (4)$$

$$= (\hat{\rho}_m - \hat{\rho}_f) \hat{\sigma}_{u_m} + \hat{\rho}_f (\hat{\sigma}_{u_m} - \hat{\sigma}_{u_f}). \quad (5)$$

The decompositions derived from (4) and (5) measure the effects of gender differences in wage error term variances and correlations between unobserved errors in the selection and wage equations. Decompositions (4) and (5) correspond to standardizing on the male correlation coefficient (female wage error variance) or on the female correlation coefficient (male wage error variance).

We can consider four alternative decompositions that in effect define labor market inequity with respect to how sample selection varies across demographic groups.

The most straight forward approach is imply to identify the overall selection component as a category apart from discrimination and endowment effects:

$$\bar{Y}_m - \bar{Y}_f = \underbrace{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f)}_{\text{discrimination}} + \underbrace{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m}_{\text{endowments}} + \underbrace{(\hat{\theta}_m \hat{\lambda}_m - \hat{\theta}_f \hat{\lambda}_f)}_{\text{selectivity}}. \quad (6)$$

If one believed that gender differences in the probit selection parameters for employment represented discrimination and that gender differences in personal attributes that determine the probability of employment are simply endowment differences, the resulting decomposition would be:

$$\begin{aligned} \bar{Y}_m - \bar{Y}_f &= \underbrace{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) + \hat{\theta}_m (\hat{\lambda}_m^0 - \hat{\lambda}_f)}_{\text{discrimination}} \\ &+ \underbrace{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m + \hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f^0)}_{\text{endowments}} \\ &+ \underbrace{(\hat{\theta}_m - \hat{\theta}_f) \hat{\lambda}_f}_{\text{selectivity}}. \end{aligned} \quad (7)$$

A second alternative is to add the effects of gender differences in ρ to the estimated endowment (human capital) effects on the grounds that the gender difference in the

error correlation coefficient is a justifiable structural source of gender wage gaps. It is difficult to know where to assign the wage gap effects of gender differences in the wage error variances. Differences in wage dispersion might or might not reflect direct labor market discrimination. Consequently, we include wage dispersion effects in the neutral category of selection effects. Upon standardizing on the male wage error variance, the overall wage decomposition becomes

$$\begin{aligned}
\bar{Y}_m - \bar{Y}_f &= \underbrace{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) + \hat{\theta}_m (\hat{\lambda}_f^0 - \hat{\lambda}_f)}_{\text{discrimination}} & (8) \\
&+ \underbrace{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m + \hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f) + (\hat{\rho}_m - \hat{\rho}_f) \hat{\sigma}_{u_f}}_{\text{endowments}} \\
&+ \underbrace{\hat{\rho}_m (\hat{\sigma}_{u_m} - \hat{\sigma}_{u_f})}_{\text{selectivity}}.
\end{aligned}$$

The most encompassing view of discrimination is to regard both gender differences in the estimated γ parameters from the probit selection equation for employment and gender differences in the wage effects of selectivity (θ) as manifestations of discrimination. Gender differences in the values of the employment determining variables (H') continue be treated as nondiscriminatory endowment effects. These assumptions lead to

$$\begin{aligned}
\bar{Y}_m - \bar{Y}_f &= \underbrace{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) + \hat{\theta}_m (\hat{\lambda}_f^0 - \hat{\lambda}_f) + (\hat{\theta}_m - \hat{\theta}_f) \hat{\lambda}_f}_{\text{discrimination}} & (9) \\
&+ \underbrace{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m + \hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f^0)}_{\text{endowments}}. \\
&= \underbrace{\bar{X}'_f (\hat{\beta}_m - \hat{\beta}_f) + \hat{\theta}_m \hat{\lambda}_f^0 - \hat{\theta}_f \hat{\lambda}_f}_{\text{discrimination}} \\
&+ \underbrace{(\bar{X}_m - \bar{X}_f)' \hat{\beta}_m + \hat{\theta}_m (\hat{\lambda}_m - \hat{\lambda}_f^0)}_{\text{endowments}}.
\end{aligned}$$

An example taken from the Israeli 1995 Census illustrates how much difference it

Wage Decompositon with Selectivity Correction (log) wage gap = 0.2567			
<u>Decomposition Method</u>	<u>Endow.</u>	<u>Disc.</u>	<u>Select.</u>
Standard (1)	0.0916 (35.68%)	0.1651 (64.32%)	0.0000 (0.00%)
Selection (6)	0.0976 (38.02%)	0.1730 (67.39%)	-0.0139 (-5.41%)
Selection (7)	0.1595 (62.13%)	0.1305 (50.84%)	-0.0333 (-12.97%)
Selection (9)	0.1595 (62.13%)	0.0972 (37.87%)	0.0000 (0.00%)

FIG. 2.

makes when selectivity is not taken into account and when it is taken into account, how much difference the chosen decomposition method makes.

The decompositions are applied to a sample of male and female professional workers. The unadjusted (log) wage differential is 0.2567. The endowment effect accounts from 35.68% to 62.13% of the wage gap, and the discrimination effect accounts from 37.87% to 67.39% of the wage gap.