Gauss-Seidel Algorithm Example

Endogenous Variables: $y_{it}, \quad i = 1, \ldots, 5$

Exogenous Variables: $Z_{it}, \quad i = 1, 2, 3$

Recursive Endogenous Variables: $y_{it}, \quad i = 1, 2$

Simultaneous Endogenous Variables: y_{it},
$i = 3, 4, 5$

Recursive Block

$$y_{1t} = f_1(Z_{2t}, Z_{3t})$$

$$y_{2t} = f_2(y_{1t}, Z_{1t})$$

Simultaneous Block

$$y_{3t} = f_3(y_{1t}, y_{4t}, y_{5t-1}, Z_{2t})$$

$$y_{4t} = f_4(y_{3t}, y_{4t-1}, Z_{1t}, Z_{3t-1})$$

$$y_{5t} = f_5(y_{3t}, y_{4t})$$

Period $T+1$ Forecast

$$\tilde{y}_{1T+1} = f_1(Z_{2T+1}, Z_{3T+1})$$

$$\tilde{y}_{2T+1} = f_2(\tilde{y}_{1T+1}, Z_{1T+1})$$

$$\tilde{y}^{(1)}_{3T+1} = f_3\left(\tilde{y}_{1T+1}, \tilde{y}^{(0)}_{4T+1}, y_{5T}, Z_{2T+1}\right)$$

$$\tilde{y}^{(1)}_{4T+1} = f_4\left(\tilde{y}^{(1)}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T}\right)$$

$$\tilde{y}^{(1)}_{5T+1} = f_5\left(\tilde{y}^{(1)}_{3T+1}, \tilde{y}^{(1)}_{4T+1}\right)$$

(typically, $\tilde{y}^{(0)}_{4T+1} = y_{4T}$)

Subsequent iterations apply only to the simultaneous block:

$$\tilde{y}^{(2)}_{3T+1} = f_3\left(\tilde{y}_{1T+1}, \tilde{y}^{(1)}_{4T+1}, y_{5T}, Z_{2T+1}\right)$$

$$\tilde{y}^{(2)}_{4T+1} = f_4\left(\tilde{y}^{(2)}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T}\right)$$

$$\tilde{y}^{(2)}_{5T+1} = f_5\left(\tilde{y}^{(2)}_{3T+1}, \tilde{y}^{(2)}_{4T+1}\right)$$

etc.

If convergence is reached on the *jth* iteration, the forecast for period $T+1$ is given by

$\tilde{Y}'_{T+1} = (\tilde{y}_{1T+1}, \tilde{y}_{2T+1}, \tilde{y}_{3T+1}, \tilde{y}_{4T+1}, \tilde{y}_{5T+1})$ where

$$\tilde{y}_{1T+1} = f_1(Z_{2T+1}, Z_{3T+1})$$

$$\tilde{y}_{2T+1} = f_2(\tilde{y}_{1T+1}, Z_{1T+1})$$

$$\tilde{y}_{3T+1} = f_3\left(\tilde{y}_{1T+1}, \tilde{y}^{(j-1)}_{4T+1}, y_{5T}, Z_{2T+1}\right)$$

$$\tilde{y}_{4T+1} = f_4(\tilde{y}_{3T+1}, y_{4T}, Z_{1T+1}, Z_{3T})$$

$$\tilde{y}_{5T+1} = f_5(\tilde{y}_{3T+1}, \tilde{y}_{4T+1})$$

$$\left(\tilde{y}_{iT+1} = \tilde{y}^{(j)}_{iT+1} \cong \tilde{y}^{(j-1)}_{iT+1}, \quad i = 3,4,5\right)$$

Period $T+2$ Forecast

$$\tilde{y}_{1T+2} = f_1(Z_{2T+2}, Z_{3T+2})$$

$$\tilde{y}_{2T+2} = f_2(\tilde{y}_{1T+2}, Z_{1T+2})$$

$$\tilde{y}^{(1)}_{3T+2} = f_3\left(\tilde{y}_{1T+2}, \tilde{y}^{(0)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2}\right)$$

$$\tilde{y}^{(1)}_{4T+2} = f_4\left(\tilde{y}^{(1)}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1}\right)$$

$$\tilde{y}^{(1)}_{5T+2} = f_5\left(\tilde{y}^{(1)}_{3T+2}, \tilde{y}^{(1)}_{4T+2}\right)$$

(typically, $\tilde{y}^{(0)}_{4T+2} = \tilde{y}_{4T+1}$)

Subsequent iterations apply only to the simultaneous block:

$$\tilde{y}^{(2)}_{3T+2} = f_3\left(\tilde{y}_{1T+2}, \tilde{y}^{(1)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2}\right)$$

$$\tilde{y}^{(2)}_{4T+2} = f_4\left(\tilde{y}^{(2)}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1}\right)$$

$$\tilde{y}^{(2)}_{5T+2} = f_5\left(\tilde{y}^{(2)}_{3T+2}, \tilde{y}^{(2)}_{4T+2}\right)$$

etc.

If convergence is reached on the jth iteration, the forecast for period $T+2$ is given by

$\tilde{Y}'_{T+2} = (\tilde{y}_{1T+2}, \tilde{y}_{2T+2}, \tilde{y}_{3T+2}, \tilde{y}_{4T+2}, \tilde{y}_{5T+2})$ where

$$\tilde{y}_{1T+2} = f_1(Z_{2T+2}, Z_{3T+2})$$

$$\tilde{y}_{2T+2} = f_2(\tilde{y}_{1T+2}, Z_{1T+2})$$

$$\tilde{y}_{3T+2} = f_3\left(\tilde{y}_{1T+2}, \tilde{y}^{(j-1)}_{4T+2}, \tilde{y}_{5T+1}, Z_{2T+2}\right)$$

$$\tilde{y}_{4T+2} = f_4(\tilde{y}_{3T+2}, \tilde{y}_{4T+1}, Z_{1T+2}, Z_{3T+1})$$

$$\tilde{y}_{5T+2} = f_5(\tilde{y}_{3T+2}, \tilde{y}_{4T+2})$$

$$\left(\tilde{y}_{iT+2} = \tilde{y}^{(j)}_{iT+2} \cong \tilde{y}^{(j-1)}_{iT+2}, \quad i = 3,4,5\right)$$

In general a forecast for period $T + \ell$ is given by

$$\tilde{Y}'_{T+\ell} = (\tilde{y}_{1T+\ell}, \tilde{y}_{2T+\ell}, \tilde{y}_{3T+\ell}, \tilde{y}_{4T+\ell}, \tilde{y}_{5T+\ell})$$ where

$$\tilde{y}_{1T+\ell} = f_1(Z_{2T+\ell}, Z_{3T+\ell})$$

$$\tilde{y}_{2T+\ell} = f_2(\tilde{y}_{1T+\ell}, Z_{1T+\ell})$$

$$\tilde{y}_{3T+\ell} = f_3\left(\tilde{y}_{1T+\ell}, \tilde{y}^{(j-1)}_{4T+\ell}, \tilde{y}_{5T+\ell-1}, Z_{2T+\ell}\right)$$

$$\tilde{y}_{4T+\ell} = f_4(\tilde{y}_{3T+\ell}, \tilde{y}_{4\bar{T}+\ell-1}, Z_{1T+\ell}, Z_{3T+\ell-1})$$

$$\tilde{y}_{5T+\ell} = f_5(\tilde{y}_{3T+\ell}, \tilde{y}_{4T+\ell})$$

$$\left(\tilde{y}_{iT+\ell} = \tilde{y}^{(j)}_{iT+\ell} \cong \tilde{y}^{(j-1)}_{iT+\ell}, \quad i = 3, 4, 5\right)$$

Gauss-Seidel Example

(1) $Y_1 = a_0 + a_1 Y_2$

(2) $Y_2 = b_0 + b_1 Y_1$

Solution:

(3) $Y_1^* = \dfrac{a_0 + a_1 b_0}{1 - a_1 b_1}$,

(4) $Y_2^* = \dfrac{b_0 + b_1 a_0}{1 - a_1 b_1}$ for $a_1 b_1 \neq 1$.

Gauss-Seidel solution method

For the jth iteration the model is specified as

(5) $Y_1^{(j)} = a_0 + a_1 Y_2^{(j-1)}$

(6) $Y_2^{(j)} = b_0 + b_1 Y_1^{(j)}$.

Upon substitution of (5) into (6) we have

(7) $\boxed{Y_2^{(j)} = b_0 + b_1 a_0 + a_1 b_1 Y_2^{(j-1)}}$

which is a first-order linear difference equation.

Accordingly, the solution to (7) is

(8) $\boxed{Y_2^{(j)} = \left[Y_2^{(0)} - \dfrac{b_0 + b_1 a_0}{1 - a_1 b_1}\right](a_1 b_1)^j + \dfrac{b_0 + b_1 a_0}{1 - a_1 b_1}}$

$$\lim_{j \to \infty} Y_2^{(j)} = \frac{b_0 + b_1 a_0}{1 - a_1 b_1} \quad (\text{for } |a_1 b_1| < 1)$$

$$= Y_2^*.$$

Alternatively, the solution algorithm can be expressed as

(9) $Y_2^{(j)} = b_0 + b_1 Y_1^{(j-1)}$

(10) $Y_1^{(j)} = a_0 + a_1 Y_2^{(j)}$

Upon substitution of (9) into (10) we have

(11) $\boxed{Y_1^{(j)} = a_0 + a_1 b_0 + a_1 b_1 Y_1^{(j-1)}}$

which is a first-order linear difference equation.

Accordingly, the general solution to (11) is

(12) $\boxed{Y_1^{(j)} = \left\{ Y_1^{(0)} - \frac{a_0 + a_1 b_0}{1 - a_1 b_1} \right\} (a_1 b_1)^j + \frac{a_0 + a_1 b_0}{1 - a_1 b_1}}$

where

$$\lim_{j \to \infty} Y_1^{(j)} = \frac{a_0 + a_1 b_0}{1 - a_1 b_1} \quad (\text{for } |a_1 b_1| < 1)$$

$$= Y_1^*.$$

Suppose $|a_1 b_1| > 1$. In this case (8) and (12) will not converge. A simple re-normalization of (1) and (2) will solve the problem:

$$Y_2 = b_0' + b_1' Y_1$$

$$Y_1 = a_0' + a_1' Y_2$$

where $b_0' = -a_0 / a_1$, $b_1' = 1 / a_1$, $a_0' = -b_0 / b_1$, and $a_1' = 1 / b_1$.

It is easily seen that $|a_1'b_1'| = |1/a_1 b_1| < 1$.

Numerical example

Let $a_0 = 25$, $a_1 = 1.5$, $b_0 = -22$, $b_1 = 0.8$ (note that $|a_1 b_1| = 1.2 > 1$).

The solution values are $Y_1^* = 40$, $Y_2^* = 10$.

The Gauss-Seidel solution method can be implemented by substituting the appropriate parameter values into (5) and (6) to obtain

$$Y_1^{(j)} = 25 + 1.5Y_2^{(j-1)} \text{ and}$$

$$Y_2^{(j)} = -22 + 0.8Y_1^{(j)} \text{ ;}$$

or by making the appropriate parameter substitutions in (7) to obtain

(13) $Y_2^{(j)} = -2 + 1.2Y_2^{(j-1)}$.

Let $Y_2^{(0)} = 0$ as an initial condition, then appropriate substitution into (8) yields

(14) $\boxed{Y_2^{(j)} = (-10)(1.2)^j + 10}$

The first-order linear difference equation corresponding to $Y_1^{(j)}$ is obtained by making the appropriate parameter substitutions in (11):

(15) $Y_1^{(j)} = -8 + 1.2Y_1^{(j-1)}$.

Given the initial condition $Y_2^{(0)} = 0$, implies the initial condition $Y_1^{(0)} = 27.5$. Accordingly, appropriate substitution into (12) yields the general solution to (15):

(16) $$Y_1^{(j)} = (-12.5)(1.2)^j + 40$$

j	$Y_1^{(j)}$	$Y_2^{(j)}$
0	27.5	0
1	25	-2
2	22	-4.4
3	18.4	-7.28
4	14.08	-10.736
.	.	.
.	.	.
.	.	.
∞	$-\infty$	$-\infty$