Econ 522a

Unconditional Forecasting Variances

For the linear model $Y_f = X_f\beta + u_f$, the unconditional forecast of Y_f is

$$\hat{Y}_f = \hat{X}_f\hat{\beta}$$

where $E(\hat{X}_f) = X_f$ (non random),$E(\hat{\beta}) = \beta$, and $\hat{X}_f$ and $\hat{\beta}$ are independently distributed. The unconditional forecast error is given by

$$\begin{aligned} e_f &= Y_f - \hat{Y}_f \\ &= X_f\beta - \hat{X}_f\hat{\beta} + u_f \end{aligned}$$

$$\begin{aligned} Var\left(e_f\right) &= Var\left(X_f\beta - \hat{X}_f\hat{\beta}\right) + Var\left(u_f\right) + \underbrace{2Cov\left[\left(X_f\beta - \hat{X}_f\hat{\beta}\right), u_f\right]}_{0} \\ &= Var\left(\hat{X}_f\hat{\beta}\right) + \sigma_u^2 \\ &= \beta'\Sigma_{\hat{X}_f}\beta + X_f\Sigma_{\hat{\beta}}X_f' + \sigma_u^2 \ + \ \text{tr}\ \left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_f}\right] \\ &= \beta'\Sigma_{\hat{X}_f}\beta + \sigma_u^2\left[X_f\left(X'X\right)^{-1}X_f'\right] + \sigma_u^2 + \ \text{tr}\ \left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_f}\right] \\ &= \beta'\Sigma_{\hat{X}_f}\beta + \sigma_u^2\left[X_f\left(X'X\right)^{-1}X_f' + 1\right] + \ \text{tr}\ \left[\Sigma_{\hat{\beta}}\cdot\Sigma_{\hat{X}_f}\right], \end{aligned}$$

where $\Sigma_{\hat{\beta}}$ and $\Sigma_{\hat{X}_f}$ are the variance/covariance matrices for $\hat{\beta}$ and $\hat{X}_f$.[1] In practice one would replace unknown values by estimated values:

$$\hat{\sigma}_{e_f}^2 = \hat{\beta}'\hat{\Sigma}_{\hat{X}_f}\hat{\beta} + \hat{\sigma}_u^2\left[\hat{X}_f\left(X'X\right)^{-1}\hat{X}_f' + 1\right] + \ \text{tr}\ \left[\hat{\Sigma}_{\hat{\beta}}\cdot\hat{\Sigma}_{\hat{X}_f}\right]$$

[1] Gerald G. Brown and Herbert C. Rutemiller, "Means and Variances of Stochastic Vector Products with Applications to Random Linear Models," *Management Science*, Vol. 24, No.2, October 1977.