Bivariate Normal Distribution

$$f(u_1,u_2)=\frac{1}{2\pi\sigma_1\sigma_2\left[1-(\rho_{12})^2\right]^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left[1-(\rho_{12})^2\right]^{-1}\left[\left(\tfrac{u_1}{\sigma_1}\right)^2-2\rho_{12}\tfrac{u_1u_2}{\sigma_1\sigma_2}+\left(\tfrac{u_2}{\sigma_2}\right)^2\right]\right\}$$

Let the variance covariance matrix for (u_1,u_2) be given by $V=\begin{bmatrix}(\sigma_1)^2 & \sigma_{12}\\ \sigma_{21} & (\sigma_2)^2\end{bmatrix}$.,

The determinant of V is given by

$$\begin{aligned}|V| &= (\sigma_1)^2(\sigma_2)^2-\sigma_{12}\sigma_{21}\\ &= (\sigma_1)^2(\sigma_2)^2\left[1-(\rho_{12})^2\right]\end{aligned}$$

so that $|V|^{\frac{1}{2}}=\sigma_1\sigma_2\left[1-(\rho_{12})^2\right]^{\frac{1}{2}}$.

Note that $V^{-1}=\begin{bmatrix}\dfrac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}} & -\dfrac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}}\\ -\dfrac{\sigma_{21}}{\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}} & \dfrac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}}\end{bmatrix}$

$$=\begin{bmatrix}\dfrac{\sigma_2^2}{(\sigma_1)^2(\sigma_2)^2\left[1-(\rho_{12})^2\right]} & -\dfrac{\sigma_{12}}{(\sigma_1)^2(\sigma_2)^2\left[1-(\rho_{12})^2\right]}\\ -\dfrac{\sigma_{21}}{(\sigma_1)^2(\sigma_2)^2\left[1-(\rho_{12})^2\right]} & \dfrac{\sigma_1^2}{(\sigma_1)^2(\sigma_2)^2\left[1-(\rho_{12})^2\right]}\end{bmatrix}$$

$$=\left[1-(\rho_{12})^2\right]^{-1}\begin{bmatrix}\dfrac{1}{(\sigma_1)^2} & -\dfrac{\sigma_{12}}{(\sigma_1)^2(\sigma_2)^2}\\ -\dfrac{\sigma_{21}}{(\sigma_1)^2(\sigma_2)^2} & \dfrac{1}{(\sigma_2)^2}\end{bmatrix}=\left[1-(\rho_{12})^2\right]^{-1}\begin{bmatrix}\dfrac{1}{(\sigma_1)^2} & -\dfrac{\rho_{12}}{\sigma_1\sigma_2}\\ -\dfrac{\rho_{12}}{\sigma_1\sigma_2} & \dfrac{1}{(\sigma_2)^2}\end{bmatrix}.$$

Finally, $\begin{pmatrix}u_1 & u_2\end{pmatrix}V^{-1}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\left[1-(\rho_{12})^2\right]^{-1}\left[\left(\dfrac{u_1}{\sigma_1}\right)^2-2\rho_{12}\dfrac{u_1u_2}{\sigma_1\sigma_2}+\left(\dfrac{u_2}{\sigma_2}\right)^2\right].$

So we can write $f(u_1,u_2)=\dfrac{1}{2\pi|V|^{\frac{1}{2}}}\exp\left(-\dfrac{1}{2}u'\ V^{-1}u\right)$ where $u'=\begin{pmatrix}u_1 & u_2\end{pmatrix}$.