Deflection calculations:

Beam: Clear view is integration method

\[ w = \frac{dV}{dx} \]
\[ V = \frac{dM}{dx} \] from equilibrium

\[ \Theta = \frac{d\phi}{dx} \] from compatibility

The link between these is flexural rigidity
\[ \phi = \frac{M}{EI} \]

Four Integration Method

Identity loads: w, v, m

\[ V = Sw_0 \, dx \]
\[ M = SVM_0 + SS_0 \, dx \]
\[ \phi = \frac{M}{EI} = \frac{1}{EI} \, SS_0 \, dx \]
\[ \phi = \frac{1}{EI} \, SS_0 \, dx \]

Boundary Conditions:
\[ d(0) = 0 \]
\[ \phi(0) = 0 \]
\[ \phi(L) = 0 \]

Max deflection:
\[ f(x) = \left[ \frac{1}{E} \right] \left[ \frac{1}{EI} \right] \left[ \frac{1}{EI} \right] \left[ \frac{1}{EI} \right] \]

\[ f(L) = \frac{W L^4}{8EI} \]
Notice on the last one we could have saved a lot of time if we go straight to the moment diagram.

Recall \( M(x) = \frac{1}{2} \bar{w} x^2 + C_1 x + C_2 \)

We found \( C_1 = -\bar{w} L \), \( C_2 = \frac{1}{2} \bar{w} L^2 \)

Thus \( M(x) = \frac{1}{2} \bar{w} x^2 - \bar{w} L x + \frac{1}{2} \bar{w} L^2 \)

\[ = \frac{1}{2} \bar{w} \left[ L^2 - 2Lx + x^2 \right] \]

\[ = \frac{\bar{w} L}{2} (L-x)^2 \]

- So we could just find \( M \) dyadically & double integrate.

For example

\( \delta(x) = 0 \)

\( \theta(x) = 0 \)

\( B.C. (1) \quad C_E = 0 \)

\( B.C. (2) \quad C_1 = 0 \)

\( \delta_{max} = \delta (L) = \frac{P}{E I} \left( \frac{L^3}{3} - \frac{L^3}{6} \right) = \frac{P L^3}{3E I} \)
A lot easier (and quicker) is energy, i.e., \( W = We \)  

\( W = \sum \delta e \cdot dW \)

For beams (bending)

\[ S_{W1} = (M_o \frac{d}{2}) (\phi \theta) \cdot dW \]

For regular beam, area is same

\[ S = \int A \cdot \frac{M_o}{x} \cdot dx \]

\[ \int \frac{M_o}{x} \cdot dx = \int M_o (I) \cdot dx = \int M_o \frac{d}{x} \cdot dx \]

All these are constant at one section

\[ W_e = W_i \]

\[ P \cdot d = \int M_o \frac{d}{x} \cdot dx \]

NOW:

\[ P \cdot d = \frac{1}{E I} \int (PL - P) \cdot dx \]

\[ P \cdot d = \frac{P^2}{E I} \int (L - x) \cdot dx \]

\[ d = \frac{P}{E I} \left( \left[ \frac{(L - x)^3}{3} \right] - 0 \right) \]

\[ = \frac{PL^3}{3EI} \]

Pretty short
Not bad. But what are the shortcomings? You can only get the deflection at the load!!

**Virtual Work**

Someone figured out the loads causing $M$ and the defects causing $\Phi$ don't have to cancel each other!!

As long as $P_0$ and $M_0$ are in equilibrium $\Phi$ don't have to cancel each other! O.K.!

**Method:** Use the real deformations, but put dummy loads where you need them.

\[
1 \cdot \Delta = \int m \left( \frac{M}{EI} \right) \, dx
\]

\[
\Delta = \int_0^l \frac{M}{EI} \, dx = \frac{W L^3}{48 EI}
\]

**REAL**

\[ M \]

\[ \Phi = \frac{M_0}{EI} \]

\[ \Delta = \frac{W L^4}{2 EI} \]

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Also:

- Doesn't work for deflection
- Can't get rotation (and shear)

\*Dummy loads only make them unit loads.
Virtual Work for trusses (truss example)
Find horizontal deflection \( u \)

**Reactions**

\[ \begin{align*}
\sum M_A &= 0 \\
100(E) - R_F (12) &= 0 \\
R_F &= 66.7 \text{ kN} \\
\sum F_U &= 0 \\
R_A + R_F &= 100 \\
R_A &= 33.3 \text{ kN} \\
\end{align*} \]

**Equations**

\[ \begin{align*}
\Delta &= \frac{1}{E I} \sum F L_i \\
\Delta &= \frac{1}{AE} \sum f \left( \frac{F L_i}{AE} \right) \\
A_{\Delta} &= \frac{L}{AE} \left[ (1)(47) + (1)(-44) + (1)(39) \right] \\
\end{align*} \]
Indeterminate Structures: Remove redundant

\[ \text{What is } M_0? \]

\[ \text{too many reactions} \]

\[ \text{But we know } \Delta_0 = 0 \]

\[ \text{for real structure} \]

\[ \Delta_{w0} - \Delta_{w0} = 0 \]

\[ \therefore \text{solve for } R \]

Then use statics.

\[ \Delta_{w0} = \frac{R h}{AE} \]

\[ \Delta_{w} = \frac{R h}{AE} \]

\[ \Delta_{w} - \Delta_{R} = \Delta_{ew} \]

\[ \therefore \text{solve for } R \]

\[ \Delta_{w} = \frac{wL^4}{8EI} \]

\[ \Delta_{R} = \frac{R L^3}{8EI} \]