1 Friction

Friction force may be imposed between contacting bodies to oppose their relative motion. Friction force can be a function of the material of the contacting bodies, the shape and the roughness of the surfaces, relative velocity, normal force, lubrication, and other factors. Therefore, determining the force caused by all possible form of friction in one formula is not possible.

In general we know friction either as Coulomb or viscous, or a combination of the two. Consider the two bodies shown in Figure 1 being in contact. Let us assume that each body applies a normal force $f_N$ on the other body causing a pair of friction forces, $(f)$, opposing the relative motion. Coulomb friction, also known as the dry friction, may exist even if the two surfaces do not move relative to each other. If the relative motion between the two bodies is zero, the Coulomb friction is computed as

$$ (f) = \mu_s f_N $$ (1)

where $\mu_s$ is the coefficient of static friction. When the relative motion is nonzero, the friction force is determined as

$$ (f) = \mu_d f_N $$ (2)

where $\mu_d$ is the coefficient of dynamic (or kinetic) friction (normally $\mu_d < \mu_s$). Assuming that the dynamic friction is not a function of the relative velocity, and that the transition from static to dynamic friction is instantaneous, Eqs. (1) and (2) can be presented graphically as shown in Figure 2(a).

Viscous friction, also known as the wet or lubricated friction, is proportional to the relative velocity of the contacting surfaces. This type of friction can be described in its simplest form as

$$ (f) = \mu_v v_{i,j} $$ (3)

where $\mu_v$ is the coefficient of viscous friction, and $v_{i,j}$ is the magnitude of the relative velocity $\vec{v}_{i,j} = \vec{v}_i - \vec{v}_j$, or relative speed. If the coefficient of viscous friction is not a constant and is a function of the normal force, then the viscous friction force can be determined as

$$ (f) = k_v v_{i,j} f_N $$ (4)

where $k_v$ is a different definition of the coefficient of viscous friction.

If the friction is assumed to be a combination of Coulomb and viscous frictions, the combined models can be presented as shown in Figure 2(b), where as the relative velocity increases so does the friction force.
It has been found experimentally that the transition of friction force from zero to nonzero relative velocity is not instantaneous, but it takes place during a short period of time. This transition, known as the Stribeck effect, is depicted in Figure 2(c).

Implementation of Coulomb and viscous frictions, including the Stribeck effect, in the equations of motion of a multibody system is not difficult as long as the relative velocity of the contacting bodies is not zero. The difficulty arises when the relative velocity becomes zero, or during the so-called stiction, the static friction force that should be \(-\mu_s f_N\leq f \leq \mu_s f_N\) cannot easily be determined in a dynamic analysis. To remedy this problem, the static friction force can be assumed to vary from \(-\mu_s f_N\) to \(\mu_s f_N\) during a short range of varied relative velocity in the vicinity of stiction, as shown in Figure 2(d).

A variety of analytical formulas to closely represent the force-velocity characteristics of the function of Figure 2(d) can be found in literature. One example of such functions is the following [1]:

\[
\begin{align*}
^{(f)}f &= f_N \left[ \mu_s + (\mu_s - \mu_d) e^{-\left(\frac{v_j}{v_s}\right)^p} \right] \tanh(k_r v_{ij}) + ^{(f)}f
\end{align*}
\]

The general shape of this function is shown in Figure 3. The viscous friction in this formula can be defined as either Eq. (3) or Eq. (4).

In addition to the coefficients of friction and the relative speed, Eq. (5) also contains other parameters. Each of these parameters can change the shape of the function. The coefficient of sliding speed, \(v_s\), changes the shape of the decay in the Stribeck region as shown in Figure 4(a), where a smaller value for this parameter results in a sharper drop from the static to dynamic friction. The exponent \(p\) affects the drop from static to dynamic friction in a different form as shown in Figure 4(b). The parameter \(k_r\) adjusts the slope of the curve from zero relative speed the maximum static friction, as depicted in Figure 4(c).
Figure 3: Friction force versus relative speed as described by Eq. (5).

Figure 4: Effects of three parameters on the friction force curve.

Typical values for the three parameters are $v_i = 0.001$ m/sec, $p = 2$, and $k_t = 10,000$. A large value for $k_t$ makes the slope of the curve in the static friction region almost vertical. We should also note that with this friction model the maximum value of the static friction force does not reach the desired value $\mu_s f_N$. Therefore we may assign a larger value for $\mu_s$ to compensate for this shortcoming of the model.

Another example of an analytical formula that closely represents the force-velocity characteristics of the function of Figure 3 is the following [2]:

$$f = f_N \mu_d \tanh(4v_r) + f_N (\mu_s - \mu_d) \frac{16v_r}{(v_r^2 + 3)^2} + \mu_s v_{ij} \tanh(4 \frac{f_N}{f_m})$$

(6)

where $v_r = \frac{v_{ij}}{v_t}$, and $v_t$ is the transition velocity indicating the velocity at which the friction force reaches its maximum value and the transition from static to dynamic region begins. The parameter $f_m$, which is called the transition force, specifies the required minimum normal force for viscous friction to take place. Typical representations of this model for different values of the parameter $v_t$ are shown in Figure 5. We note that the transition from static to dynamic friction occurs at the specified transition velocity, at which the friction force is has reached the limit of the static force $\mu_s f_N$. 

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If the normal force \( f_N \) is the result of a contact represented as a penetration model, the friction force should be categorized as an applied load. However, if the contact is modeled as a constraint, for which a Lagrange multiplier represents the resulting contact force, the corresponding friction must be categorized as a reaction force. In either case, the contact and friction models can be considered between bodies of different shapes in the contact area, as shown in Figure 6 for a few examples.

![Figure 5: Typical friction force-speed curve for the model of Eq. (6).](image)

![Figure 6: Examples of contact with or without friction between different objects.](image)

### 2 Contact Force (Continuous Analysis)

In the continuous analysis method, it is assumed that when two bodies collide, although the contact period is very small, the change in the velocities is not discontinuous—the velocities vary continuously during the period of contact as the contacting bodies undergo local deformations. To represent the local deformations in a simplified manner, it is assumed that the colliding bodies penetrate into each other in the contact region as depicted in the examples of Figure 7(a). The amount of penetration, which is denoted as \( \delta \), represents the sum of local deformations of the two contacting bodies. The penetration results in a pair of resistive contact forces acting on the two bodies in opposite directions as shown in the illustrations of Figure 7(b). The penetration starts in a compression phase with \( \delta = 0 \) and reaches a maximum value \( \delta = \delta_m \). During the restitution phase, the penetration returns from \( \delta = \delta_m \) back to \( \delta = 0 \). Following an impact, the two bodies may remain in contact or they may separate.

A contact force model can be viewed as a logical point-to-point spring-damper that is only active during the period of contact. The characteristics of the logical spring-damper element must be adjusted...
based on the material properties of the contacting bodies. In the following subsections, we consider several models for a body contacting a rigid surface, and for two bodies contacting each other.

Figure 7 (a) Penetration models for contact-impact, and (b) the corresponding FBD’s.

2.1 A Body Contacting A Rigid Surface

The simplest contact model to consider is the mass-spring system shown in Figure 8. The single body with the mass $m$ moves toward a rigid surface. As the body contacts the surface it deforms (or penetrates the surface) by an amount $\delta$. Assuming the deformation is represented by a spring of stiffness $k$, the penetration force can be determined as

$$f_N = k\delta$$  \hspace{1cm} (7)

If we assume that the deformation also contains damping, the penetration force can be computed as

$$f_N = k\delta + d_c\dot{\delta}$$  \hspace{1cm} (8)

where $d_c$ is the damping coefficient.

Figure 8 A mass-spring contact model.

In the models of Eqs. (7) and (8) it is assumed that both the spring and the damper have linear characteristics. We may also consider nonlinear characteristics for the spring as

$$f_N = k\delta^n$$  \hspace{1cm} (9)

where $n > 1$ is an exponent to be determined based on the material characteristics of the body. With such a nonlinear model for the spring, Eq. (8) becomes:

$$f_N = k\delta^n + d_c\dot{\delta}$$  \hspace{1cm} (10)

In this nonlinear model, a reasonable value for the exponent is $n = 3/2$ as it will be discussed in the next subsection.
In the model of Eq. (7) the penetration force vary linearly during both compression and restitution phases as shown as a solid line in Figure 9(a). The force starts from zero at \( \delta = 0 \), reaches a maximum value, and then returns along the same straight line back to zero, where at this point the body separates from the contacting surface. When we add damping to the model, as in Eq. (8), the paths of the compression and restitution phases separate as shown in dashed line in Figure 9(a). This curve shows that as the body contacts the surface, there is a nonzero penetration force due to the velocity of the body at the start of the compression phase. This velocity causes a penetration force \( f_N = d_v (-\dot{\delta}) \), where \( (-\dot{\delta}) \) denotes the initial penetration speed at contact. The figure shows that the path during the restitution phase, in the presence of damping, is different from that of the compression phase (called hysteresis). We also observe that near the end of the restitution phase, the penetration force becomes zero, at \( \dot{\delta} = \delta_c \), before the restitution phase is completed. This is the instant at which the body separates from the surface while it continues to undeform. At this point, since \( f_N = 0 \), the acceleration of the body is zero and, therefore, the velocity of the body is at a maximum. This means that during the final portion of the restitution phase, the model yields a tensile force, as shown in the figure in dotted line, which should not be applied to the body.

The contact force versus penetration for the model of Eq. (9) is shown in solid line in Figure 9(b). Similar to the linear model, in the absence of damping, this model provides the same paths for both compression and restitution phases. When damping is included, as in Eq. (10), the model exhibits hysteresis behavior as shown in dashed line. Similar to the linear model, the penetration force becomes zero at \( \delta = \delta_s \), which is the instant at which the body leaves the contacting surface.

\[ k = \frac{k_1 k_2}{k_1 + k_2} \]  

Figure 9 Compression and restitution phases for (a) the linear and (b) the nonlinear models.

### 2.2 Two-Body Contact

The discussion of the previous subsection can be extended to two bodies becoming in contact. Assume that the two bodies shown in Figure 10 are in the process of contacting each other. The force models of Eqs. (7) through (10) can be implemented between the two bodies to determine the contact force. The overall penetration, \( \delta \), can be considered as the sum of the deformations of the two bodies as \( \delta = \delta_1 + \delta_2 \). The penetration is determined along the axis of contact, which is normal to the contacting surfaces.

The stiffness \( k \), for example in the linear model of Eq. (7), can be determined based on the stiffness of the two springs as

\[ k = \frac{k_1 k_2}{k_1 + k_2} \]
In the presence of damping, as in Eq. (8) and Eq. (10), the two bodies separate at \( \delta = \delta_1 \) before they have completely recovered their deformations, as demonstrated in Figure 9.

Figure 10 Two bodies becoming in contact.

The spring model with the nonlinear characteristics of Eq. (9) is known as the Hertz contact force model [3]. The model considers the stiffness to depend on the material property and the local geometry of the contacting bodies. For example, for two colliding spheres with radii \( R_i \) and \( R_j \), the parameter \( k \) can be determined as

\[
k = \frac{4}{3\pi(h_i + h_j)} \left( \frac{R_i R_j}{R_i + R_j} \right)^2; \quad h_i = \frac{1 - \nu_k^2}{\pi E_k}; \quad k = i, j
\]  

(12)

where \( \nu_k \) and \( E_k \) are the Poisson’s ratio and Young’s modulus of each sphere, and the exponent in Eq. (9) is determined to be \( n = 3/2 \). If one of the contacting surfaces is flat (very large radius of \( R_j = \infty \)), Eq. (12) becomes

\[
k = \frac{4R_i^2}{3\pi(h_i + h_j)}; \quad h_i = \frac{1 - \nu_k^2}{\pi E_k}; \quad k = i, j
\]  

(13)

A revised form of damping for Eq. (6) considers the damping coefficient to be a function of indentation as [4]

\[
d_c = \mu\delta^n
\]  

(14)

where \( \mu \) is called the hysteresis damping factor. Substituting this description of the damping coefficient in Eq. (10) results in the following contact force model:

\[
f_N = (k + \mu\delta)\delta^n
\]  

(15)

Figure 11 illustrates a comparison between this contact force model (solid line) and the force model of Eq. (10). In the revised model, the contact forces at the start of the compression phase and at the end of the restitution phase are zero, which is quite different from the original model of Eq. (10). In the revised model, since the contact force does not reach zero until at the end of the restitution phase, the bodies do not separate until the restitution is fully completed.
As has been shown in Eq. (14) can be expressed as a function of the coefficient of restitution, where such a function cannot be found analytically for the damping coefficient in Eq. (10). The coefficient of restitution is defined as the ratio between the relative penetration speeds at the start of the compression phase, $e_0$, and at the instant of separation, $e_1$:

$$e = \frac{e_1}{e_0}$$  \hspace{1cm} (16)

One description of the damping factor in term of the coefficient of restitution has been suggested as [5]

$$\mu = \frac{3k(1-e^2)}{4e_0}$$  \hspace{1cm} (17)

Substitution of this expression in Eq. (15) provides the following contact force model:

$$f_N = k\delta^e + \frac{3(1-e^2)}{4e_0}$$  \hspace{1cm} (18)

A graphical description of this force model is shown in Figure 12. For values of the coefficient of restitution smaller than 1.0, the force-penetration curves exhibit hysteresis behavior. The area inside each hysteresis curve denotes the loss of energy during impact. For an elastic impact, that is $e=1.0$, the loss of energy is zero. In other words, the relative speed at the moment of separation is equal to the relative speed at the start of contact. Note that Eq. (18) assumes that during both the compression and restitution phases, $\delta > 0$, where the sign of $\dot{\delta}$ is positive during compression (in the same direction as that of $\delta$) and negative during restitution.
Figure 12 Contact force of Eq. (18) versus penetration for different values of the coefficient of restitution.

Typical values for the stiffness parameter $k$ are $(1 \rightarrow 10) \times 10^{10}$ N/m$^{1.5}$. The coefficient of restitution can have a value $0 < e < 1.0$. For $e = 1.0$, Eq. (18) becomes identical to Eq. (9); that is, no energy loss due to damping. This represents an elastic impact causing the separation speed to be equal to the approach speed. In contrast $e = 0$, representing an inelastic (or plastic) impact, results in the maximum loss of energy due to damping.

The model of Eq. (18) provides a better representation of contact for values of $e$ closer to “1.0” than the values closer to “0”. This shortcoming of Eq. (18) has been improved upon in the following force model [6, 7]:

$$f_N = K\delta^n\left[1 + \frac{8(1-e)}{5e} \frac{\dot{\delta}}{\ddot{\delta}}\right]$$

(19)

In this model, the damping factor becomes larger, compared to that of Eq. (18), as the coefficient of restitution becomes smaller. We note that for $e = 0$, the damping factor becomes infinity and, therefore, for computational purposes the coefficient of restitution should not be set exactly to zero but it could be a very small number.

References