Appendix A

Euler Angles and Bryant Angles

Among the most common parameters used to describe the angular orientation of a body in space are *Euler angles*. The angular orientation of a given body-fixed coordinate system $\xi\eta\zeta$ can be envisioned to be the result of three successive rotations. The three angles of rotation corresponding to the three successive rotations are defined as Euler angles. The sequence of rotations used to define the final orientation of the coordinate system is to some extent arbitrary. A total of twelve conventions is possible in a right-hand coordinate system. For the Euler angles described here, a particular sequence of rotations known as the *x convention* is considered. Another convention, known as the *xyz convention,* is also discussed here; the parameters associated with this convention are often referred to as *Bryant angles*.

A.1 Euler Angles

Euler angles provide a set of three coordinates without any constraint equations. The sequence of rotations employed in the x convention starts by rotating the initial system of xyz axes counterclockwise about the z axis by an angle ψ, as shown in Fig. A.1. The resulting coordinate system is labeled $\xi''\eta''\zeta''$. In the second step the intermediate $\xi''\eta''\zeta''$ axes are rotated about ξ'' counterclockwise by an angle θ to produce another intermediate set, the $\xi'\eta'\zeta'$ axes. Finally, the $\xi'\eta'\zeta'$ axes are rotated counterclockwise about ζ' by an angle σ to produce the desired $\xi\eta\zeta$ system of axes.[†] The angles ψ, θ, and σ, which are the Euler angles, completely specify the orientation of the $\xi\eta\zeta$ system relative to the xyz system and can therefore act as a set of three independent coordinates.

[†]In most textbooks, the third Euler angle is denoted by ϕ. Since, in this text, ϕ is used to describe the angle of rotation about the orientational axis of rotation, σ is used here for the third Euler angle instead of ϕ.

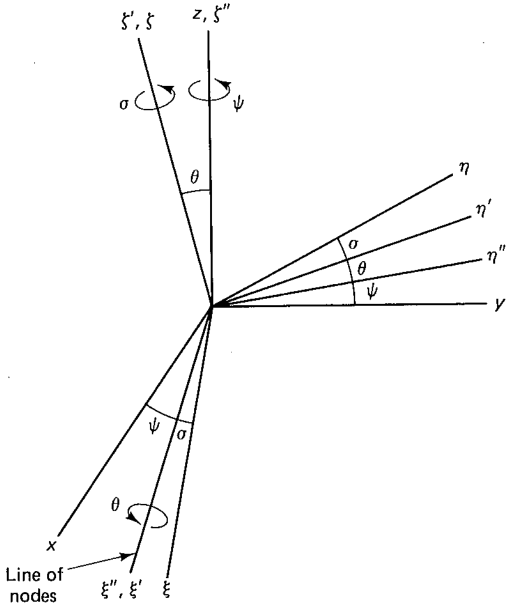

Figure A.1 The rotations defining the Euler Angles.

The elements of the complete transformation matrix **A** can be obtained as the triple product of the matrices that define the separate rotations, i.e., the matrices

$$\mathbf{D} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} c\sigma & -s\sigma & 0 \\ s\sigma & c\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where c ≡ cos and s ≡ sin. Hence, $\mathbf{A} = \mathbf{DCB}$ is found to be

$$\mathbf{A} = \begin{bmatrix} c\psi c\sigma - s\psi c\theta s\sigma & -c\psi s\sigma - s\psi c\theta c\sigma & s\psi s\theta \\ s\psi c\sigma + c\psi c\theta s\sigma & -s\psi s\sigma + c\psi c\theta c\sigma & -c\psi s\theta \\ s\theta s\sigma & s\theta c\sigma & c\theta \end{bmatrix} \tag{A.1}$$

It can be verified that matrix **A** is orthonormal, i.e., that $\mathbf{A}^T = \mathbf{A}^{-1}$.

The advantage of having three independent rotational coordinates, instead of nine dependent direction cosines, is offset by the disadvantage that the elements of **A** in terms of the Euler angles are complicated trigonometric functions. Still, a more severe problem exists. Figure A.2 shows that if $\theta = n\pi, n = 0, \pm 1, \pm 2, \ldots$, the axes of the first and third rotations coincide, so that ψ and σ cannot be distinguished. This fact is illustrated by setting $\theta = 0$ in **A** to obtain

$$\mathbf{A} = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & -c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta = 0$$

where $\alpha = \psi + \sigma$.

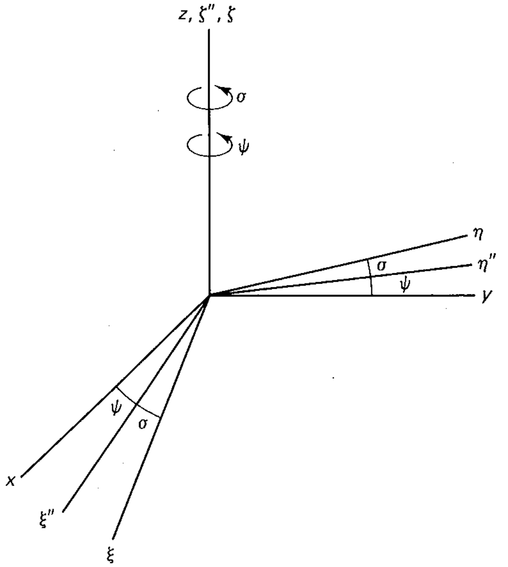

Figure A.2 Euler angles for the case $\theta = n\pi$ ($n = 0, \pm 1, \pm 2, \dots$).

It may be necessary to calculate Euler angles that correspond to a known transformation matrix. For this purpose, the following formulas are deduced from Eq. A.1:

$$\begin{aligned} \cos\theta &= a_{33} & \sin\theta &= \pm\sqrt{1-\cos^2\theta} \\ \cos\psi &= \frac{-a_{23}}{\sin\theta} & \sin\psi &= \frac{a_{13}}{\sin\theta} \\ \cos\sigma &= \frac{a_{32}}{\sin\theta} & \sin\sigma &= \frac{a_{31}}{\sin\theta} \end{aligned} \tag{A.2}$$

These formulas show that numerical difficulties are to be expected for values of θ that are close to the critical values $n\pi$, $n = 0, \pm 1, \pm 2, \dots$.

A.1.1 Time Derivatives of Euler Angles

The general rotation associated with $\vec{\omega}$ can be considered equivalent to three successive rotations with angular velocities $\omega_{(\psi)} = \dot{\psi}$, $\omega_{(\theta)} = \dot{\theta}$, and $\omega_{(\sigma)} = \dot{\sigma}$. Hence, the vector $\vec{\omega}$ can be obtained as the sum of three separate angular velocity vectors. This vector sum cannot be obtained easily, since the directions $\vec{\omega}_{(\psi)}$, $\vec{\omega}_{(\theta)}$, and $\vec{\omega}_{(\sigma)}$ are not orthogonally placed: $\vec{\omega}_{(\psi)}$ is along the global z axis and, $\vec{\omega}_{(\theta)}$ is along the line of nodes, while $\vec{\omega}_{(\sigma)}$ is along the body ζ axis. However, the orthonormal transformation matrices **B**, **C**, and **D** may be used to determine the components of these vectors along any desired set of axes.

Figure A.3 can be used to obtain the components of the velocity vector $\vec{\omega}$ in the $\xi\eta\zeta$ axes in terms of Euler angles and rates. Since $\dot{\psi}$ is parallel to the z axis, its components along the body axes are given by applying the orthonormal transformation $\mathbf{B}^T\mathbf{C}^T$.

$$\begin{aligned} \dot{\psi}_{(\xi)} &= \dot{\psi}\sin\theta\sin\sigma \\ \dot{\psi}_{(\eta)} &= \dot{\psi}\sin\theta\cos\sigma \\ \dot{\psi}_{(\zeta)} &= \dot{\psi}\cos\theta \end{aligned}$$

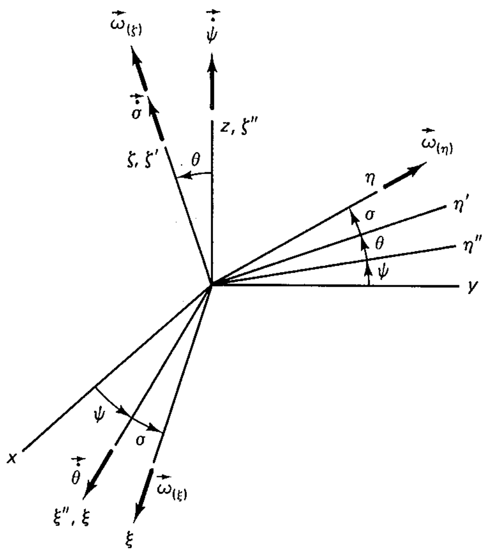

Figure A.3 Euler angle rates.

The line of nodes, which is the direction of $\vec{\dot{\theta}}$, coincides with the ξ' axis, and so the components of $\vec{\dot{\theta}}$ with respect to the body axes are furnished by applying only the final orthonormal transformation $\mathbf{B}^T$:

$$\dot{\theta}_{(\xi)} = \dot{\theta} \cos \sigma$$
$$\dot{\theta}_{(\eta)} = -\dot{\theta} \sin \sigma$$
$$\dot{\theta}_{(\zeta)} = 0$$

No transformation is necessary for the component of $\vec{\dot{\sigma}}$, which lies along the ζ axis. When these components of the separate angular velocities are added, the components of $\vec{\omega}$ with respect to the body axes are

$$\omega_{(\xi)} = \dot{\psi} \sin \theta \sin \sigma + \dot{\theta} \cos \sigma$$
$$\omega_{(\eta)} = \dot{\psi} \sin \theta \cos \sigma - \dot{\theta} \sin \sigma$$
$$\omega_{(\zeta)} = \dot{\psi} \cos \theta + \dot{\sigma}$$

or, in matrix form,

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \sin \theta \sin \sigma & \cos \sigma & 0 \\ \sin \theta \cos \sigma & -\sin \sigma & 0 \\ \cos \theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\sigma} \end{bmatrix} \tag{A.3}$$

In addition, the Euler angle rates can be expressed in terms of $\omega_{(\xi)}$, $\omega_{(\eta)}$, and $\omega_{(\zeta)}$. Since Euler angle rates are not orthogonal, the inverse of the matrix of Eq. A.3 yields

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\sigma} \end{bmatrix} = \frac{1}{\sin \theta} \begin{bmatrix} \sin \sigma & \cos \sigma & 0 \\ \cos \sigma \sin \theta & -\sin \sigma \sin \theta & 0 \\ -\sin \sigma \cos \theta & -\cos \sigma \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} \tag{A.4}$$

Similar techniques may be applied to express the components of $\vec{\omega}$ along the xyz axes, in terms of Euler angles and rates. Equation A.4 shows, again, that numerical problems will arise if θ is close to the critical values $n\pi$, $n = 0, \pm 1, \ldots$.

A.2 BRYANT ANGLES[19]

The Bryant angle convention considers rotations about axes other than those for the Euler angles. The first rotation may be carried out counterclockwise about the x axis through an angle ϕ_1; the resultant coordinate system will be labeled $\xi''\eta''\zeta''$, as shown in Fig. A.4. The second rotation, through an angle ϕ_2 counterclockwise about the η'' axis, produces the coordinate system $\xi'\eta'\zeta'$. Finally, the third rotation, counterclockwise about the ζ' axis through an angle ϕ_3, results in the $\xi\eta\zeta$ coordinate system. The transformation matrices for the individual rotations are

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi_1 & -s\phi_1 \\ 0 & s\phi_1 & c\phi_1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} c\phi_2 & 0 & s\phi_2 \\ 0 & 1 & 0 \\ -s\phi_2 & 0 & c\phi_2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} c\phi_3 & -s\phi_3 & 0 \\ s\phi_3 & c\phi_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, the matrix of the complete transformation, $\mathbf{A} = \mathbf{DCB}$, is

$$\mathbf{A} = \begin{bmatrix} c\phi_2 c\phi_3 & -c\phi_2 s\phi_3 & s\phi_2 \\ c\phi_1 s\phi_3 + s\phi_1 s\phi_2 c\phi_3 & c\phi_1 c\phi_3 - s\phi_1 s\phi_2 s\phi_3 & -s\phi_1 c\phi_2 \\ s\phi_1 s\phi_3 - c\phi_1 s\phi_2 c\phi_3 & s\phi_1 c\phi_3 + c\phi_1 s\phi_2 s\phi_3 & c\phi_1 c\phi_2 \end{bmatrix} \tag{A.5}$$

Again, it may be necessary to calculate Bryant angles that correspond to a known transformation matrix. This can be done, with the help of formulas derived from

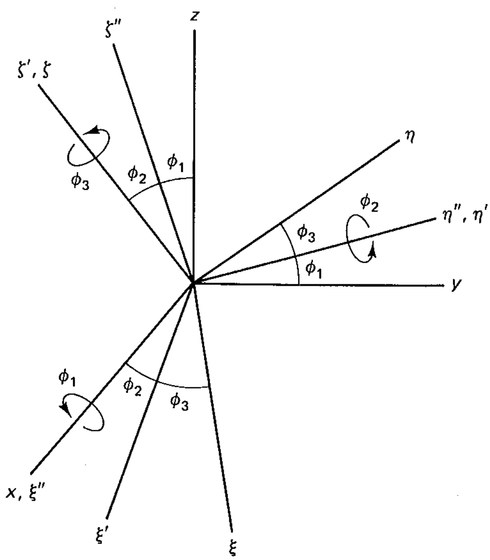

Figure A.4 Rotations defining Bryant angles.

Eq. A.5, to obtain

$$\begin{aligned} \sin\phi_2 &= a_{13} & \cos\phi_2 &= \pm\sqrt{1-\sin^2\phi_2} \\ \sin\phi_1 &= -\frac{a_{23}}{\cos\phi_2} & \cos\phi_1 &= \frac{a_{33}}{\cos\phi_2} \\ \sin\phi_3 &= -\frac{a_{12}}{\cos\phi_2} & \cos\phi_3 &= \frac{a_{11}}{\cos\phi_2} \end{aligned} \tag{A.6}$$

It can be observed again that there exists a critical case, namely, when $\phi_2 = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, in which the axes of the first and third rotations coincide, so that the rotation angles ϕ_1 and ϕ_3 become indistinguishable.

A.2.1 Time Derivative of Bryant Angles

The relationship between angular velocity $\vec{\omega}$ and Bryant angles and rates can be found in a similar fashion to that for the Euler rates. The transformation matrix for the velocity components is

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \cos\phi_1\cos\phi_3 & \sin\phi_3 & 0 \\ -\cos\phi_2\sin\phi_3 & \cos\phi_3 & 0 \\ \sin\phi_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \end{bmatrix} \tag{A.7}$$

The inverse transformation can be found to be

$$\begin{bmatrix} \dot\phi_1 \\ \dot\phi_2 \\ \dot\phi_3 \end{bmatrix} = \frac{1}{\cos\phi_2} \begin{bmatrix} \cos\phi_3 & -\sin\phi_3 & 0 \\ \sin\phi_3\cos\phi_2 & \cos\phi_3\cos\phi_2 & 0 \\ -\cos\phi_3\sin\phi_2 & \sin\phi_3\sin\phi_2 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} \tag{A.8}$$

It can be seen that Eq. A.8 fails numerically in the vicinity of the critical values $\phi_2 = \pi/2 + n\pi, n = 0, +1, \ldots$.

Appendix B

Relationship between Euler Parameters and Euler Angles

In some kinematics problems, the angular orientation of a body with respect to the global coordinate system is described in terms of Euler angles and it is desired to determine the corresponding set of Euler parameters, or vice versa. There are simple formulas that can be used directly to find one set of variables if the other set is known.

B.1 EULER PARAMETERS IN TERMS OF EULER ANGLES

If the angular orientation of a local coordinate system is described in terms of three Euler angles ψ, θ, and σ, it is possible to find the corresponding Euler parameters. The trace of matrix $\mathbf{A}$ in terms of Euler angles, from Eq. A.1, is

$$\operatorname{tr} \mathbf{A} = 4 \cos^2 \frac{\theta}{2} \cos^2 \frac{\psi + \sigma}{2} - 1$$

Then, Eq. 6.25 yields

$$e_0 = \cos \frac{\theta}{2} \cos \frac{\psi + \sigma}{2} \tag{B.1}$$

From Eqs. 6.26a–c with a_{11}, a_{22}, and a_{33} taken from the transformation matrix of Eq. A.1, it is found that

$$e_1 = \sin \frac{\theta}{2} \cos \frac{\psi - \sigma}{2} \tag{B.2}$$

$$e_2 = \sin \frac{\theta}{2} \sin \frac{\psi - \sigma}{2} \tag{B.3}$$

$$e_3 = \cos \frac{\theta}{2} \sin \frac{\psi + \sigma}{2} \tag{B.4}$$

Note that the four Euler parameters can always be determined if the three Euler angles are known.

B.2 EULER ANGLES IN TERMS OF EULER PARAMETERS

The Euler angles can be determined from the Euler parameters by comparing the transformation matrices in terms of Euler angles and Euler parameters: i.e., Eqs. A.1 and 6.19. Equating the a_{33} elements of the two matrices yields

$$\cos\theta = 2(e_0^2 + e_3^2) - 1 \tag{B.5}$$

Equating a_{32} and a_{23}, we get

$$\cos\sigma = \frac{2(e_2e_3 + e_0e_1)}{\sin\theta} \tag{B.6}$$

$$\cos\psi = -\frac{2(e_2e_3 - e_0e_1)}{\sin\theta} \tag{B.7}$$

It is clear that for $\sin\theta = 0$, $\cos\sigma$ and $\cos\psi$ cannot be evaluated. In this case, from Eq. B.5, $\cos\theta = 1$ yields

$$e_0^2 + e_3^2 = 1 \tag{B.8}$$

Then, from the constraints between Euler parameters, i.e., Eq. 6.21, it is found that

$$e_1^2 + e_2^2 = 0 \tag{B.9}$$

which can be true only if

$$e_1 = e_2 = 0 \tag{B.10}$$

Since $\mathbf{e} = [e_1, e_2, e_3]^T$ consists of the components of $\vec{e}$ along both the xyz and $\xi\eta\zeta$ coordinate axes, Eq. B.10 indicates that the orientational axis of rotation, denoted by $\vec{e}$, is along the z or the ζ axis. The ambiguity for $\theta = k\pi, k = 0, 1, \ldots,$ is discussed in Appendix A in more detail. However, if the a_{21} elements of the two transformation matrices are used when $\cos\theta = 1$, it is found that

$$\sin(\psi + \sigma) = 2e_0e_3 \tag{B.11}$$

Now, if either ψ or σ is given an arbitrary value, the value of the other can be determined.

Appendix C

Coordinate Partitioning with L-U Factorization

Crout's algorithm LU-I from Sec. 3.3.3 can easily be modified to perform L-U factorization on nonsquare matrices. If L-U factorization with full pivoting is performed on an $m \times n$ matrix $\mathbf{A}$, it may result in the following partitioned form:

$$m\left\{\underbrace{\begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix}}_{n}\right. \longrightarrow \begin{array}{r} m-s\Big\{ \\ s\{ \\ \\ \end{array}\begin{array}{cc} \left[\begin{array}{c|c} \mathbf{L}\diagdown\mathbf{U} & \mathbf{R} \\ \hline \mathbf{S} & \mathbf{D} \end{array}\right] \\ \underbrace{\qquad}_{m-s}\ \underbrace{\quad}_{n-m+s} \end{array}$$

It is assumed that there are s redundant rows in the matrix that have ended up as the bottom s rows after factorization as a result of full pivoting. The rank of this matrix is $m - s$. The $\mathbf{L}$ and $\mathbf{U}$ matrices occupy the $(m - s) \times (m - s)$ top left elements, and $\mathbf{D}$ is a submatrix all of whose elements begin at approximately zero (i.e., smaller than a specified tolerance). The left $m - s$ columns of the factored matrix are called the basic columns, and the remaining $n - m + s$ columns are the nonbasic columns. If all of the rows of $\mathbf{A}$ are independent, i.e., if $s = 0$, then L-U factorization with full or partial (column) pivoting partitions $\mathbf{A}$ as follows:

$$m\left\{\underbrace{\begin{bmatrix} & & \\ & \mathbf{A} & \\ & & \end{bmatrix}}_{n}\right. \longrightarrow m\left\{\begin{array}{c} \left[\begin{array}{c|c} \mathbf{L}\diagdown\mathbf{U} & \mathbf{R} \end{array}\right] \\ \underbrace{\qquad}_{m}\ \underbrace{\qquad}_{n-m} \end{array}\right.$$

Without any loss of generality, it can be assumed that $\mathbf{A}$ represents the Jacobian matrix $\mathbf{\Phi_q}$, where all of the m constraints are independent. Since the elements of $\mathbf{q}$ correspond to the column indices of $\mathbf{\Phi}_q$, the indices of the columns of **L** (or **U**) define the dependent (basic) coordinates **u**, and indices of the columns of **R** define the independent (nonbasic) coordinates **v**.[18]

Partitioning of **q** into **u** and **v** also corresponds to the partitioning of $\mathbf{\Phi}_q$ into $\mathbf{\Phi}_u$ and $\mathbf{\Phi}_v$. In terms of the **L**, **U**, and **R** matrices,

$$\mathbf{\Phi}_u = \mathbf{LU} \tag{C.1}$$

$$\mathbf{\Phi}_v = \mathbf{LR} \tag{C.2}$$

In some well-developed L-U factorization subroutines, matrix **R** is replaced by a matrix **H**, as follows:

$$\left[\begin{array}{cc|c} \ddots & \mathbf{U} & \\ \mathbf{L} & \ddots & \mathbf{R} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \ddots & \mathbf{U} & \\ \mathbf{L} & \ddots & \mathbf{H} \end{array}\right]$$

where

$$\mathbf{H} = -\mathbf{U}^{-1}\mathbf{R} \tag{C.3}$$

This yields

$$\begin{aligned} \mathbf{\Phi}_v &= -\mathbf{LUH} \\ &= -\mathbf{\Phi}_u\mathbf{H} \end{aligned}$$

or

$$\mathbf{H} = -\mathbf{\Phi}_u^{-1}\mathbf{\Phi}_v \tag{C.4}$$

The matrix **H** is called the *influence coefficient matrix*. This matrix relates variations of **u** to variations of **v**. This is obtained by taking the differential of the constraint equations $\mathbf{\Phi} = \mathit{0}$:

$$\delta\mathbf{\Phi} = \mathbf{\Phi}_q\,\delta\mathbf{q} = \mathit{0}$$

or

$$\mathbf{\Phi}_u\,\delta\mathbf{u} + \mathbf{\Phi}_v\,\delta_v = \mathit{0}$$

which yields

$$\begin{aligned} \delta\mathbf{u} &= -\mathbf{\Phi}_u^{-1}\mathbf{\Phi}_v\,\delta\mathbf{v} \\ &= \mathbf{H}\,\delta\mathbf{v} \end{aligned} \tag{C.5}$$

The kinematic velocity equations also yield

$$\dot{\mathbf{u}} = \mathbf{H}\dot{\mathbf{v}} \tag{C.6}$$