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Static Equilibrium Analysis

Transient dynamic analysis of complex mechanical systems is often initiated from a position of static equilibrium. Assigning correct values to the coordinates that describe a state of static equilibrium can be a complicated (almost impossible) task for large-scale interconnected systems of bodies. Therefore, static equilibrium analysis is often performed to find the correct set of coordinates prior to the dynamic analysis. As an example, consider a spatial multibody system representing a particular vehicle. The model contains elements representing the main chassis, the wheels, and the suspension and steering systems. The springs of the suspension system may have nonlinear characteristics, and so may the stiffness of the tires. Finding the coordinates describing the static equilibrium configuration of this system from the available data and figures, or even from the actual vehicle, is not a trivial task.

In this chapter several methods for static equilibrium analysis are presented. These methods are based on the general formulation of the governing equations of motion given in Eqs. 13.12 through 13.15. The static equilibrium equations presented in this chapter can easily be modified to fit any particular formulation or any set of coordinates.

14.1 AN ITERATIVE METHOD

Since velocities and accelerations are zero for static equilibrium, Eqs. 13.12 to 13.15 yield the equilibrium equations as

$$\boldsymbol{\Phi}(\mathbf{q}) = \mathbf{0} \tag{14.1}$$

$$\boldsymbol{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} + \mathbf{g} = \mathbf{0} \tag{14.2}$$

This is a set of $m + n$ nonlinear algebraic equations in $m + n$ unknowns $\boldsymbol{\lambda}$ and $\mathbf{q}$. An iterative technique such as Newton-Raphson can be employed to solve Eqs. 14.1 and 14.2. The corresponding iterative equation is

$$\begin{bmatrix} \boldsymbol{\Phi}_\mathbf{q} & \mathbf{0} \\ (\boldsymbol{\Phi}_\mathbf{q}^T\boldsymbol{\lambda} + \mathbf{g})_\mathbf{q} & \boldsymbol{\Phi}_\mathbf{q}^T \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q} \\ \Delta\boldsymbol{\lambda} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Phi}_\mathbf{q}^T\boldsymbol{\lambda} + \mathbf{g} \end{bmatrix} \tag{14.3}$$

This formula requires proper initial estimates for $\mathbf{q}$ and $\boldsymbol{\lambda}$. Determination of a reasonable set of initial estimates for $\mathbf{q}$ is possible. However, a proper set of initial estimates for $\boldsymbol{\lambda}$ is rather difficult. A poor estimate for $\boldsymbol{\lambda}$ may lead to a badly conditioned matrix and divergence of the algorithm. This is an undesirable feature of this method.

It is possible to obtain an estimate for $\boldsymbol{\lambda}$ by solving Eq. 13.16 for $t = 0$. The initial estimates for $\mathbf{q}$, with $\dot{\mathbf{q}} = \mathbf{0}$, are used to solve Eq. 13.16 for $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$. Then, vector $\boldsymbol{\lambda}$ obtained from this solution might be a reasonable estimate to start the iterative process of Eq. 14.3.

14.1.1 Coordinate Partitioning

In order to circumvent the difficulty of finding a reasonable initial estimate for $\boldsymbol{\lambda}$ in Eq. 14.3, the coordinate partitioning method can eliminate the presence of Lagrange multipliers in the equilibrium equations.[18] If $\mathbf{q}$ is partitioned into m dependent coordinates $\mathbf{u}$ and k independent coordinates $\mathbf{v}$, then $\boldsymbol{\Phi}_\mathbf{q}$ and $\mathbf{g}$ can be partitioned as follows:

$$\boldsymbol{\Phi}_\mathbf{q} \equiv [\boldsymbol{\Phi}_\mathbf{u}, \boldsymbol{\Phi}_\mathbf{v}]$$

$$\mathbf{g} = \begin{bmatrix} \mathbf{g}_{(u)} \\ \mathbf{g}_{(v)} \end{bmatrix}$$

Hence, Eq. 14.2 is written as

$$\boldsymbol{\Phi}_\mathbf{u}^T\boldsymbol{\lambda} + \mathbf{g}_{(u)} = \mathbf{0} \tag{a}$$

$$\boldsymbol{\Phi}_\mathbf{v}^T\boldsymbol{\lambda} + \mathbf{g}_{(v)} = \mathbf{0} \tag{b}$$

If Eq. a is solved for $\boldsymbol{\lambda}$ and the result is substituted in Eq. b, it is found that

$$\mathbf{g}_{(v)} = -\mathbf{H}^T\mathbf{g}_{(u)} \tag{14.4}$$

where $\mathbf{H}$ is as defined in Appendix C:

$$\mathbf{H} = -\boldsymbol{\Phi}_\mathbf{u}^{-1}\boldsymbol{\Phi}_\mathbf{v}$$

Equation 14.4 can be written as

$$\mathbf{f} \equiv \mathbf{H}^T\mathbf{g}_{(u)} + \mathbf{g}_{(v)} = \mathbf{0} \tag{14.5}$$

which represents k equations in k unknowns $\mathbf{v}$. Equation 14.5 can be solved by a Newton-Raphson algorithm provided an initial estimate is given for $\mathbf{v}$. The iterative formula in this case is

$$\left[\frac{d\mathbf{f}}{d\mathbf{v}}\right]\Delta\mathbf{v} = -\mathbf{f} \tag{14.6}$$

Finding explicit expressions for the elements of $d\mathbf{f}/d\mathbf{v}$ can be too complicated, since $\mathbf{H}$ and $\mathbf{g}$ are implicitly functions of $\mathbf{v}$. Therefore, this matrix is evaluated by a numerical

differencing method. This can be done by perturbing one element of $\mathbf{v}$ at a time to determine its corresponding column of $d\mathbf{f}/d\mathbf{v}$. If $\mathbf{v}$ is defined prior to and after perturbation of its ith element by the equations

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_k \end{bmatrix} \qquad \mathbf{v}^i = \begin{bmatrix} v_1 \\ \vdots \\ v_i + \Delta v_i \\ \vdots \\ v_k \end{bmatrix}$$

then

$$\frac{d\mathbf{f}}{dv_i} = \frac{1}{\Delta v_i}[\mathbf{f}(\mathbf{v}^i) - \mathbf{f}(\mathbf{v})] \tag{14.7}$$

This process is repeated for $i = 1, \ldots, k$ to obtain all k columns of $d\mathbf{f}/d\mathbf{v}$.

14.2 POTENTIAL ENERGY FUNCTION

For the purpose of static equilibrium analysis, the definition of potential energy is stated here without proof. For a conservative system, the vector of forces $\mathbf{g}$ can be derived from a potential energy function V such that[5]

$$\mathbf{g} = -V_{\mathbf{q}}^T \tag{14.8}$$

The potential energy V depends only on position coordinates $\mathbf{q}$ and not on time or velocities. The most common factors contributing to the system potential energy are gravitational field, externally applied forces, and springs.

Assume that the gravitational field is acting on a system in the negative z direction. If the gravitational constant is g and the mass of body i is m_i, then Eq. 14.8 is written as

$$-m_i g = -\frac{\partial V_i^{(g)}}{\partial z} \tag{a}$$

where $V_i^{(g)}$ denotes the potential energy of body i due to gravity. Eq. a yields

$$V_i^{(g)} = m_i g(z_i - z^0) \tag{14.9}$$

where z^0 is a constant. For simplification, the potential energy $V^{(g)}$ may be taken as zero at the xy plane. Then Eq. 14.9 becomes

$$V_i^{(g)} = m_i z_i g \tag{14.10}$$

If a constant external force $\vec{f}_k$ acts on body i, the potential energy corresponding to $\vec{f}_k$ can be written as

$$V_i^{(f)} = -(f_{(x)}x_i + f_{(y)}y_i + f_{(z)}z_i) \tag{14.11}$$

where, as was done with the gravitational force, the xyz coordinate system is considered the reference frame in which the potential energy due to $\vec{f}_k$ is zero in the coordinate planes.

Potential (strain) energy stored in the kth translational spring, which connects bodies i and j, may be written as

$$V_i^{(s)} = \int_{l^0}^{l} f^{(s)}\, dl \tag{b}$$

If the spring characteristic is linear, then $f^{(s)} = k(l - l^0)$ yields

$$V_i^{(s)} = \tfrac{1}{2}k(l - l^0)^2 \tag{14.12}$$

If the spring has nonlinear characteristics and the force $f^{(s)}$ is available as a nonlinear function of Δl, then Eq. b may be integrated numerically to obtain the potential energy.

Contributions to the potential energy of a system from other sources, such as rotational springs or external moments, can be determined as in the preceding derivations. The total potential energy of a system may be defined as the sum of the potential energies of the springs, and other externally applied forces. If there are b bodies in the system, the total potential energy due to the gravitational field is

$$V^{(g)} = \sum_{i=1}^{b} V_i^{(g)} \tag{14.13}$$

For c constant externally applied forces, Eq. 14.11 yields

$$V^{(f)} = \sum_{k=1}^{c} V_k^{(f)} \tag{14.14}$$

Similarly, if there are d translational springs in the system, the total potential energy of the springs is

$$V^{(s)} = \sum_{k=1}^{d} V_k^{(s)} \tag{14.15}$$

The system potential energy is thus

$$V = V^{(g)} + V^{(f)} + V^{(s)} \tag{14.16}$$

If there are any other force elements in the system that are not discussed here, their contribution to the total potential energy must be added to Eq. 14.16.

14.2.1 Minimization of Potential Energy

The static equilibrium configuration of a mechanical system may be determined by evaluating the position for which the potential energy function is at its minimum.[5] For the potential energy function $V = V(\mathbf{q})$ of Eq. 14.16, the minimization problem can be stated as

$$\text{Minimize } V = V(\mathbf{q})$$

$$\text{subject to constraints } \boldsymbol{\Phi}(\mathbf{q}) = \mathbf{0} \tag{14.17}$$

This equation represents a constrained minimization problem; i.e., the kinematic constraints $\boldsymbol{\Phi}(\mathbf{q}) = \mathbf{0}$ must be satisfied for all feasible $\mathbf{q}$. Many constrained optimization algorithms are available and can be applied to this problem. However, Eq. 14.17 can be transformed into an unconstrained minimization problem.[14] The method is based on the coordinate partitioning method.

The total differential of the potential energy function may be written as

$$\begin{aligned} dV &= V_{\mathbf{q}}\, d\mathbf{q} \\ &= V_{\mathbf{u}}\, d\mathbf{u} + V_{\mathbf{v}}\, d\mathbf{v} \\ &= (V_{\mathbf{u}}\mathbf{H} + V_{\mathbf{v}})\, d\mathbf{v} \end{aligned}$$

where Eq. C.5 of Appendix C has been employed. This equation can be written as

$$\frac{dV}{d\mathbf{v}} = V_{\mathbf{u}}\mathbf{H} + V_{\mathbf{v}} \tag{14.18}$$

Hence, the minimization problem can be restated as

Minimize $V = V(\mathbf{v})$

$$\text{where } \frac{dV}{d\mathbf{v}} = V_{\mathbf{u}}\mathbf{H} + V_{\mathbf{v}} \tag{14.19}$$

This is an unconstrained optimization problem in k variables. The gradient vector $dV/d\mathbf{v}$ is needed for most of the commonly used optimization algorithms, e.g., steepest descent or conjugate gradient.

The gradient vector $dV/d\mathbf{v}$ can be determined easily at a feasible position $\mathbf{q}$, i.e., where $\mathbf{\Phi}(\mathbf{q}) = \mathbf{0}$ is satisfied. The partial derivative $V_{\mathbf{q}}$ is the negative of the force vector $\mathbf{g}^T$ at position $\mathbf{q}$; hence,

$$V_{\mathbf{u}} = -\mathbf{g}_{(u)}^T$$

or

$$V_{\mathbf{v}} = -\mathbf{g}_{(v)}^T$$

$$\frac{dV}{d\mathbf{v}} = -\mathbf{g}_{(u)}^T\mathbf{H} - \mathbf{g}_{(v)}^T \tag{14.20}$$

The following steps outline an algorithm, based on the minimization of potential energy, for finding a stable equilibrium configuration. Any well-developed unconstrained minimization algorithm can be used.

ALGORITHM SE-1

(a) Main routine

(a.1) Specify initial estimates for $\mathbf{q}$.

(a.2) Specify (or determine automatically) the independent variables $\mathbf{v}$.

(a.3) Enter the minimization routine.

(b) Minimization routine

(b.1) In the process of minimization, the function under consideration and its vector of gradient (in this case V and $dV/d\mathbf{v}$) must be evaluated for any particular $\mathbf{v}$. This is accomplished by a call to FUNCTION routine.

(b.2) If Flag = 0, then continue the minimization process. If Flag = 1, then a feasible solution for the kinematic constraint equations for the assigned values of $\mathbf{v}$ does not exist.

One of the following two conditions may exist:

(1) The step taken by the minimization routine is too large. Make the step smaller; e.g., cut it by half, update $\mathbf{v}$, and return to step b.1 to repeat the process.

(2) The step taken by the minimization routine is too small. This may be an indication that the set of coordinates considered as independent coordinates is not an adequate set. In order to define a new set of independent coordinates return to step a.2.

(c) FUNCTION routine

(c.1) Knowing $\mathbf{v}$, solve the constraint equations for $\mathbf{u}$. If $\mathbf{u}$ cannot be found, then set Flag = 1 and return to the minimization routine. Otherwise set Flag = 0 and continue to step c.2.

(c.2) Knowing $\mathbf{u}$ and $\mathbf{v}$; i.e., $\mathbf{q}$, evaluate the potential energy function from Eq. 14.16.

(c.3) Evaluate matrices $\mathbf{\Phi}_u$ and $\mathbf{\Phi}_v$, and then $\mathbf{H}$.

(c.4) Evaluate the force vector $\mathbf{g}$ and partition it into $\mathbf{g}_{(u)}$ and $\mathbf{g}_{(v)}$.

(c.5) Compute the gradient vector $dV/d\mathbf{v}$ from Eq. 14.20.

(c.6) Return.

It is stated in step c.1 of this algorithm that for some predicted values of $\mathbf{v}$ a solution to the constraint equations $\mathbf{\Phi}(\mathbf{u}, \mathbf{v}) = \mathbf{0}$ may not exist. This can be because the unconstrained minimization algorithm is not aware of the presence of the constraint equations. The algorithm predicts the largest possible step toward a minimum along a predicted direction. The predicted point may be outside the feasible region of the constraint equations. In cases such as this, the step must be made smaller.

If the initial estimate for the coordinates is too far from the equilibrium state, then, as was true of the coordinate partitioning method in dynamic analysis, the initial set of independent coordinates may not be valid in other points. Therefore, the preceding algorithm requires occasional checking or possibly switching to a different set of independent coordinates.

14.3 FICTITIOUS DAMPING METHOD

A mechanical system in motion with no damping elements can oscillate about its static equilibrium state. If several dampers, which are energy-dissipating elements, are added to the system, the total energy of the system will decrease as time passes. The oscillation will be slowed, and finally the mechanical system will reach its static equilibrium state. Therefore, if a mechanical system contains some damping elements, then its static equilibrium state may be determined by performing dynamic analysis, using any of the algorithms stated in Chap. 13.

For systems containing no damping elements, or not having a sufficient number of energy-dissipating elements, fictitious damping terms can be included in the equations of motion:

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} = \mathbf{g} - \mathbf{D}\dot{\mathbf{q}} \tag{14.21}$$

Matrix $\mathbf{D}$ is a positive-definite matrix containing the fictitious damping coefficients. For simplification, $\mathbf{D}$ can be defined as a diagonal matrix. The values of the fictitious damping coefficients do not change the static equilibrium state, but influence the speed of reaching that state.

14.4 JOINT COORDINATE METHOD

In Sec. 13.4 and subsections 13.4.1 and 13.4.2, a method based on the joint coordinates was introduced for the transient dynamic analysis of mechanical systems. The joint coordinates in conjunction with the potential energy function can be used for determining the static equilibrium configuration of a system.

For a system containing an open chain (single-branch or multibranch), such as the one shown in Fig. 14.1(a), a vector $\mathbf{s}$ is defined in terms of the coordinates of the base body and the joint coordinates:

$$\mathbf{s} = \begin{bmatrix} \mathbf{q}_{\text{base}} \\ \boldsymbol{\theta} \end{bmatrix} \tag{14.22}$$

For a known vector $\mathbf{s}$, vector $\mathbf{q}$ representing the configuration of the system can be computed as stated in Sec. 13.4. Then the potential energy of the system in that configuration can be computed. The static equilibrium configuration may be found by minimizing a function f:

$$\text{Minimize } f = V(\mathbf{s}) \tag{14.23}$$

where the gradient vector is

$$\begin{aligned} \frac{df}{d\mathbf{s}} &= \frac{dV}{d\mathbf{s}} \\ &= V_{\mathbf{q}} \left[\frac{d\mathbf{q}}{d\mathbf{s}} \right] \\ &= -\mathbf{g}^T \left[\frac{d\mathbf{q}}{d\mathbf{s}} \right] \end{aligned} \tag{14.24}$$

The elements of the matrix $[d\mathbf{q}/d\mathbf{s}]$ can be computed by a numerical differencing method similar to that shown in Eq. 14.7. Note that since $\mathbf{q}$ is computed directly from $\mathbf{q}_{\text{base}}$ and $\boldsymbol{\theta}$, there is no constraint violation to consider.

For systems with closed loops, such as the one shown in Fig. 14.1(b), each closed loop can be cut at one of the joints to achieve an equivalent open-chain system. For the

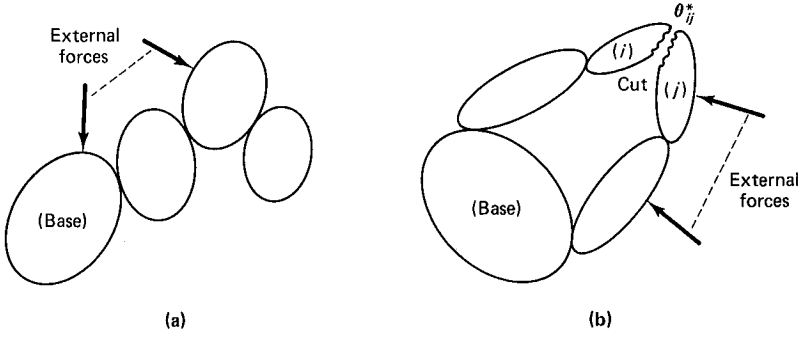

Figure 14.1 Schematic presentation of a mechanical system with (a) an open chain and (b) a closed loop.

equivalent system, vector $\mathbf{s}$ is defined as in Eq. 14.22, where $\boldsymbol{\theta}$ does not include the joint coordinates at the cut edges; such as $\boldsymbol{\theta}_{ij}^*$ in Figure 14.1(b). If the kinematic joints at the cut edges are defined by the constraint equations $\boldsymbol{\Phi}^*(\mathbf{q}) = \mathbf{0}$, then a minimization problem can be stated as

$$\text{Minimize } f = V(\mathbf{s}) + \tfrac{1}{2} w \boldsymbol{\Phi}^{*T} \boldsymbol{\Phi}^* \tag{14.25}$$

where w is a weighting coefficient. Note that in this minimization problem, the constraint violations at the cut edges are introduced in the objective function. The weighting coefficient w is introduced to scale the two terms in Eq. 14.25. Depending on the unit system used, and also the reference frame for zero potential energy, the magnitude of the potential energy V is normally several orders of magnitude larger than the sum of the squares of the constraint violations $\boldsymbol{\Phi}^{*T}\boldsymbol{\Phi}^*$. Therefore, the minimization algorithm is not sensitive to the constraint violation as much as to the changes in the potential energy. For this reason, a large value for w can bring the magnitude of both terms within the same range. The magnitude of w may be redefined several times during the minimization process.

The gradient vector for the function of Eq. 14.25 is

$$\begin{aligned} \frac{df}{d\mathbf{s}} &= \frac{dV}{d\mathbf{s}} + w\boldsymbol{\Phi}^{*T}\left[\frac{d\boldsymbol{\Phi}^*}{d\mathbf{s}}\right] \\ &= \left[-\mathbf{g}^T + w\boldsymbol{\Phi}^{*T}\boldsymbol{\Phi}_{\mathbf{q}}^*\right]\left[\frac{d\mathbf{q}}{d\mathbf{s}}\right] \end{aligned} \tag{14.26}$$