11

Spatial Dynamics

A general system of constrained equations of rigid-body spatial motion is formulated in this chapter on the basis of the principles of dynamics discussed in Chap. 8 and the spatial kinematics formulation from Chaps. 6 and 7. The equations of motion are formulated in terms of Euler parameters. The formulation developed here is identical in nature to that for planar systems in Chap. 9; the principal difference between the formulations for spatial and planar dynamics is in their dimensionality.

11.1 VECTOR OF FORCES

The forces and moments acting on a body can be due to such force elements as springs, dampers, or gravity, among others. The derivation of equations to calculate the forces (or moments) of these force elements in spatial motion is identical to that shown in Secs. 9.2.1 through 9.2.7 for planar motion. If the resultant force and moment acting on body i are $\vec{f}_i$ and $\vec{n}_i$, they must be transformed into the coordinate system in which the equations of motion are derived. For the translational equations of motion shown in Eq. 8.31, the force $\vec{f}_i$ must be defined in terms of its xyz components; i.e., $\mathbf{f}_i$. If the rotational equations of motion given by Eq. 8.32 are used, then the moment $\vec{n}_i$ must be defined in terms of its $\xi\eta\zeta$ components; i.e., $\mathbf{n}$. However, if Euler parameters are used and the equations of motion are expressed in terms of these coordinates, then the moment $\vec{n}_i$ must be transformed to a coordinate system associated with the Euler parameters.

11.1.1 Conversion of Moments

It is possible to convert the three rotational equations of motion represented by Eq. 8.32 to four rotational equations of motion associated with the Euler parameters (this will be seen in Sec. 11.2.1). In this case the moment $\vec{n}_i$ must be expressed in terms of four components denoted by $\mathbf{n}_i^*$. The objective is to find the transformation between $\mathbf{n}_i'$ (or $\mathbf{n}_i$) and $\mathbf{n}_i^*$. Two methods for deriving this transformation are shown here.

The first method is based on the scalar product of two vectors. As long as two vectors are described in the same coordinate system, their scalar product yields a scalar quantity independent of the coordinate system in which the vectors are expressed. In Eq. 8.32, $\mathbf{n}_i'$ is expressed in the same coordinate system as $\boldsymbol{\omega}_i'$. When Euler parameters are used, the moment $\mathbf{n}_i^*$ must be expressed in the same coordinate system as $\dot{\mathbf{p}}_i$. Hence,

$$\dot{\mathbf{p}}_i^T\mathbf{n}_i^* = \boldsymbol{\omega}_i'^T\mathbf{n}_i' \tag{a}$$

Then, Eq. 6.107 yields

$$\mathbf{n}_i^* = 2\mathbf{L}_i^T\mathbf{n}_i' \tag{11.1}$$

If the global components of these vectors are considered, then Eq. a is also equal to $\boldsymbol{\omega}_i^T\mathbf{n}_i$, and therefore it can be found that

$$\mathbf{n}_i^* = 2\mathbf{G}_i^T\mathbf{n}_i \tag{11.2}$$

The inverse transformations are

$$\mathbf{n}_i' = \tfrac{1}{2}\mathbf{L}_i\mathbf{n}_i^* \tag{11.3}$$

and

$$\mathbf{n}_i = \tfrac{1}{2}\mathbf{G}_i\mathbf{n}_i^* \tag{11.4}$$

The second method considers the virtual displacement of the point of application of a force on a body. In Fig. 11.1, $\vec{f}_i$ acts on point P_i and the moment of the force is $\mathbf{n}_i' = \tilde{\mathbf{s}}_i'\mathbf{f}_i'$. The position of P_i is

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i' \tag{11.5}$$

The total differential of Eq. 11.5 is

$$\begin{aligned}\delta\mathbf{r}_i^P &= \delta\mathbf{r}_i + \frac{\delta(\mathbf{A}_i\mathbf{s}_i')}{\delta\mathbf{p}_i}\,\delta\mathbf{p}_i \\ &= \delta\mathbf{r}_i + 2\mathbf{G}_i\bar{\mathbf{s}}_i'\,\delta\mathbf{p}_i + 2\mathbf{s}_i'\mathbf{p}_i^T\,\delta\mathbf{p}_i\end{aligned} \tag{b}$$

Since the four Euler parameters are subject to the constraint $\mathbf{p}_i^T\mathbf{p}_i^{-1} = 0$, the total differential of this constraint yields

$$\frac{\partial(\mathbf{p}_i^T\mathbf{p}_i - 1)}{\partial\mathbf{p}_i}\,\delta\mathbf{p}_i = 0$$

or

$$\mathbf{p}_i^T\,\delta\mathbf{p}_i = 0 \tag{11.6}$$

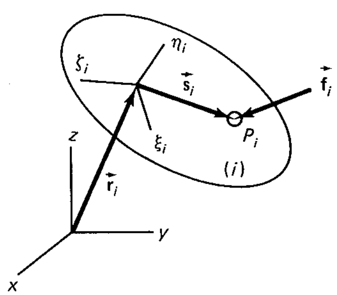

Figure 11.1 Applied forces.

Hence, Eq. b is simplified as follows:

$$\begin{aligned}\delta\mathbf{r}_i^P &= \delta\mathbf{r}_i + 2\mathbf{G}_i\bar{\mathbf{s}}_i'\,\delta\mathbf{p}_i \\ &= \delta\mathbf{r}_i + 2\mathbf{A}_i\mathbf{L}_i\bar{\mathbf{s}}_i'\,\delta\mathbf{p}_i \\ &= \delta\mathbf{r}_i + 2\mathbf{A}_i(-\tilde{\mathbf{s}}_i'\mathbf{L}_i + \mathbf{s}_i'\mathbf{p}_i^T)\,\delta\mathbf{p}_i \\ &= \delta\mathbf{r}_i - 2\tilde{\mathbf{s}}_i\mathbf{G}_i\,\delta\mathbf{p}_i\end{aligned}$$

where Eqs. 6.49, 6.71, and 6.88 have been employed.

The virtual work done by $\mathbf{f}_i$ is

$$\begin{aligned}\delta w_i &= \mathbf{f}_i^T\,\delta\mathbf{r}_i^P \\ &= \mathbf{f}_i^T(\delta\mathbf{r}_i - 2\tilde{\mathbf{s}}_i\mathbf{G}_i\,\delta\mathbf{p}_i) \\ &= \mathbf{f}_i^T\,\delta\mathbf{r}_i + 2\mathbf{n}_i^T\mathbf{G}_i\,\delta\mathbf{p}_i\end{aligned} \tag{11.7}$$

This equation shows that the virtual work δw_i is the sum of the virtual work of the force $\mathbf{f}_i$ causing a virtual translation $\delta\mathbf{r}_i$ and the virtual work of a moment $\mathbf{n}_i^* = 2\mathbf{G}_i^T\mathbf{n}_i$ causing a virtual rotation $\delta\mathbf{p}_i$. This result agrees with Eq. 11.2.

11.2 EQUATIONS OF MOTION FOR AN UNCONSTRAINED BODY

The translational equations of motion for an unconstrained body are given by Eq. 8.31 as

$$\mathbf{N}_i\ddot{\mathbf{r}}_i = \mathbf{f}_i \tag{11.8}$$

where $\mathbf{N}_i = \mathrm{diag}\,[m, m, m]_i$. The rotational equations of motion for an unconstrained body given by Eq. 8.32 are converted into three different forms in this section.

Formulation I. Substitution of Eqs. 6.107 and 6.111 into Eq. 8.32 yields

$$2\mathbf{J}_i'\mathbf{L}_i\ddot{\mathbf{p}}_i + 4\mathbf{L}_i\dot{\mathbf{L}}_i^T\mathbf{J}_i'\mathbf{L}_i\dot{\mathbf{p}}_i = \mathbf{n}_i' \tag{11.9}$$

Premultiplication of this equation by $2\mathbf{L}_i^T$ gives

$$\mathbf{J}_i^*\ddot{\mathbf{p}}_i + 2\mathbf{L}_i^T\mathbf{L}_i\mathbf{H}_i\dot{\mathbf{p}}_i = \mathbf{n}_i^* \tag{11.10}$$

where

$$\mathbf{J}_i^* = 4\mathbf{L}_i^T\mathbf{J}_i'\mathbf{L}_i \tag{11.11}$$

and

$$\mathbf{H}_i = 4\dot{\mathbf{L}}_i^T\mathbf{J}_i'\mathbf{L}_i \tag{11.12}$$

Using Eq. 6.46 and defining

$$\begin{aligned}\sigma_i &= 8\dot{\mathbf{p}}_i^T\mathbf{L}_i^T\mathbf{J}_i'\mathbf{L}_i\dot{\mathbf{p}}_i \\ &= -8\mathbf{p}_i^T\dot{\mathbf{L}}_i^T\mathbf{J}_i'\mathbf{L}_i\dot{\mathbf{p}}_i\end{aligned} \tag{11.13}$$

we can write Eq. 11.10 as

$$\mathbf{J}_i^*\ddot{\mathbf{p}}_i + \sigma_i\mathbf{p}_i + 2\mathbf{H}_i\dot{\mathbf{p}}_i = \mathbf{n}_i^* \tag{11.14}$$

This represents the rotational equations of motion in terms of $\ddot{\mathbf{p}}_i$. However, since the four Euler parameters are not independent, Eq. 6.61, i.e,

$$\mathbf{p}_i^T\ddot{\mathbf{p}} + \dot{\mathbf{p}}_i^T\dot{\mathbf{p}}_i = 0 \tag{11.15}$$

must be considered with Eq. 11.14. Equations 11.14 and 11.15 in matrix form are

$$\begin{bmatrix} \mathbf{J}^* & \mathbf{p} \\ \mathbf{p}^T & 0 \end{bmatrix}_i \begin{bmatrix} \ddot{\mathbf{p}} \\ \sigma \end{bmatrix}_i + \begin{bmatrix} 2\mathbf{H}\dot{\mathbf{p}} \\ \dot{\mathbf{p}}^T\dot{\mathbf{p}} \end{bmatrix}_i = \begin{bmatrix} \mathbf{n}^* \\ 0 \end{bmatrix}_i \tag{11.16}$$

Equation 11.16 contains five equations that can be solved for $\ddot{\mathbf{p}}_i$ and σ_i if $\mathbf{n}'_i, \mathbf{p}_i, \dot{\mathbf{p}}_i$, and $\mathbf{J}'_i$ are known. This, obviously, gives the same value for σ_i as given by Eq. 11.13! The artificial variable σ_i was defined in such a way as to have an exact inverse to the 5×5 matrix at the left in Eq. 11.16. In Eq. 11.14, σ_i can be interpreted as a Lagrange multiplier associated with the constraint equation $\mathbf{p}_i^T\mathbf{p}_i - 1 = 0$.

Formulation II. Equation 11.15 can be appended to Eq. 11.9 and written in matrix form to yield

$$\begin{bmatrix} 2\mathbf{J}'\mathbf{L} \\ \mathbf{p}^T \end{bmatrix}_i \ddot{\mathbf{p}}_i + \begin{bmatrix} \mathbf{L}\mathbf{H} \\ \dot{\mathbf{p}}^T \end{bmatrix}_i \dot{\mathbf{p}}_i = \begin{bmatrix} \mathbf{n}' \\ 0 \end{bmatrix}_i \tag{11.17}$$

If $\mathbf{n}'_i, \mathbf{p}_i, \dot{\mathbf{p}}_i$, and $\mathbf{J}'_i$ are known, then Eq. 11.17 can be solved exactly for $\ddot{\mathbf{p}}_i$. Note that the matrix at the left in Eq. 11.17 is a 4×4 matrix.

Formulation III. In the third formulation, the rotational equations of motion are left in their original form in terms of the angular velocities; i.e.,

$$\mathbf{J}'_i\dot{\boldsymbol{\omega}}'_i + \tilde{\boldsymbol{\omega}}'_i\mathbf{J}'_i\boldsymbol{\omega}'_i = \mathbf{n}'_i \tag{11.18}$$

It is clear that $\dot{\boldsymbol{\omega}}'_i$ can be calculated from this equation if $\mathbf{n}'_i, \boldsymbol{\omega}'_i$, and $\mathbf{J}'_i$ are known.

A comparison of these three formulations shows that Eq. 11.16 contains five equations in terms of $\ddot{\mathbf{p}}_i$ and σ_i, Eq. 11.17 contains four equations in terms of $\ddot{\mathbf{p}}_i$, and Eq. 11.18 contains three equations in terms of $\dot{\boldsymbol{\omega}}'_i$.

11.3 EQUATIONS OF MOTION FOR A CONSTRAINED BODY

For a constrained mechanical system containing body i, it is assumed that there are m independent constraint equations,

$$\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\mathbf{q}) = \mathbf{0} \tag{11.19}$$

where $\mathbf{q}$ contains the coordinates of all of the bodies in the system, including the coordinates of body i:

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{p}_i \end{bmatrix}$$

It was shown in Sec. 8.4.3 that the constraint reaction forces could be described in the form given by Eq. 8.50 in terms of the Jacobian matrix of the system and a vector of Lagrange multipliers:

$$\mathbf{g}^{(*)} = \boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} \tag{11.20}$$

This equation was obtained with the assumption that the vectors of forces $\mathbf{g}$ and $\mathbf{g}^{(*)}$ were defined in a coordinate system consistent with $\mathbf{q}$.

The constrained translational equations of motion for body i can be written, from Eq. 11.8, as

$$\mathbf{N}_i\ddot{\mathbf{r}}_i = \mathbf{f}_i + \mathbf{f}_i^{(c)}$$

From Eq. 11.20 it is found that

$$\mathbf{f}_i^{(c)} = \mathbf{\Phi}_{\mathbf{r}_i}^T \boldsymbol{\lambda}$$

Therefore,

$$\mathbf{N}_i \ddot{\mathbf{r}}_i - \mathbf{\Phi}_{\mathbf{r}_i}^T \boldsymbol{\lambda} = \mathbf{f}_i \qquad (11.21)$$

This represents the translational equations of motion for constrained body i. The rotational equations of motion for this body are derived in three forms corresponding to the formulations of Sec. 11.2.

Formulation I. The rotational equations of motion from Eq. 11.14 for constrained body i are written as

$$\mathbf{J}_i^* \ddot{\mathbf{p}}_i + \sigma_i \mathbf{p}_i + 2\mathbf{H}_i \dot{\mathbf{p}}_i = \mathbf{n}_i^* + \mathbf{n}_i^{*(c)}$$

Since $\mathbf{n}_i^*$ and $\mathbf{n}_i^{*(c)}$ are described in the same coordinate system as $\mathbf{p}_i$, Eq. 11.20 yields

$$\mathbf{n}_i^{*(c)} = \mathbf{\Phi}_{\mathbf{p}_i}^T \boldsymbol{\lambda}$$

Therefore,

$$\mathbf{J}_i^* \ddot{\mathbf{p}}_i + \sigma_i \mathbf{p}_i + 2\mathbf{H}_i \dot{\mathbf{p}}_i - \mathbf{\Phi}_{\mathbf{p}_i}^T \boldsymbol{\lambda} = \mathbf{n}_i^* \qquad (11.22)$$

Equations 11.22 and 11.15 are the rotational equations of motion for a constrained body.

Formulation II. Equation 11.9 is written for a constrained body as

$$2\mathbf{J}_i' \mathbf{L}_i \ddot{\mathbf{p}}_i + \mathbf{L}_i \mathbf{H}_i \dot{\mathbf{p}}_i = \mathbf{n}_i' + \mathbf{n}_i'^{(c)}$$

The transformation of moments is given by Eq. 11.3 as

$$\begin{aligned} \mathbf{n}_i'^{(c)} &= \tfrac{1}{2}\mathbf{L}_i \mathbf{n}_i^{*(c)} \\ &= \tfrac{1}{2}\mathbf{L}_i \mathbf{\Phi}_{\mathbf{p}_i}^T \boldsymbol{\lambda} \end{aligned}$$

Therefore,

$$2\mathbf{J}_i' \mathbf{L}_i \ddot{\mathbf{p}}_i + \mathbf{L}_i \mathbf{H}_i \dot{\mathbf{p}}_i - \tfrac{1}{2}\mathbf{L}_i \mathbf{\Phi}_{\mathbf{p}_i}^T \boldsymbol{\lambda} = \mathbf{n}_i' \qquad (11.23)$$

Equations 11.23 and 11.15 together can be used as the rotational equations of motion for a constrained body.

Formulation III. Equation 11.18 can be written for a constrained body as

$$\mathbf{J}_i' \dot{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_i' \mathbf{J}_i' \boldsymbol{\omega}_i' = \mathbf{n}_i' + \mathbf{n}_i'^{(c)}$$

or

$$\mathbf{J}_i' \dot{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_i' \mathbf{J}_i' \boldsymbol{\omega}_i' - \tfrac{1}{2}\mathbf{L}_i \mathbf{\Phi}_{\mathbf{p}_i}^T \boldsymbol{\lambda} = \mathbf{n}_i' \qquad (11.24)$$

In this equation the constraint equations, and hence the Jacobian matrix, are expressed in terms of Euler parameters. However, the joint reaction moments are converted to a set of coordinates consistent with $\boldsymbol{\omega}_i'$ and $\mathbf{n}_i'$.

11.4 SYSTEM OF SPATIAL EQUATIONS OF MOTION

In the preceding sections, the equations of motion for a single body were derived. Three formulations for the rotational equations of motion were shown. For a system of b bodies, constrained or unconstrained, these equations can be repeated b times in any of the three forms to find the system equations.

11.4.1 Unconstrained Bodies

For a system of b unconstrained bodies, three formulations are given.

Formulation I. Equations 11.8, 11.14, and 11.15, with a slight rearrangement, are written for all b bodies as

$$\begin{bmatrix} \mathbf{M}^* & \mathbf{P}^T \\ \mathbf{P} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\sigma} \end{bmatrix} + \begin{bmatrix} \mathbf{b}^* \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{g}^* \\ \mathit{0} \end{bmatrix} \tag{11.25}$$

where

$$\mathbf{M}^* = \begin{bmatrix} \mathbf{N}_1 & & & & \\ & \mathbf{J}_1^* & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathbf{N}_b & \\ & & & & \mathbf{J}_b^* \end{bmatrix} \tag{11.26}$$

$$\mathbf{P} = \begin{bmatrix} \mathit{0}^T & \mathbf{p}_1^T & & & \\ & & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathit{0}^T & \mathbf{p}_b^T \end{bmatrix} \tag{11.27}$$

$$\ddot{\mathbf{q}} = \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \ddot{\mathbf{p}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \ddot{\mathbf{p}}_b \end{bmatrix} \tag{11.28}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_b \end{bmatrix} \tag{11.29}$$

$$\mathbf{b}^* = \begin{bmatrix} \mathit{0} \\ 2\mathbf{H}_1\dot{\mathbf{p}}_1 \\ \vdots \\ \mathit{0} \\ 2\mathbf{H}_b\dot{\mathbf{p}}_b \end{bmatrix} \tag{11.30}$$

$$\mathbf{c} = \begin{bmatrix} \dot{\mathbf{p}}_1^T\dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{p}}_b^T\dot{\mathbf{p}}_b \end{bmatrix} \tag{11.31}$$

$$\mathbf{g}^* = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{n}_1^* \\ \vdots \\ \mathbf{f}_b \\ \mathbf{n}_b^* \end{bmatrix} \tag{11.32}$$

Formulation II. Equations 11.8 and 11.17, with a slight rearrangement, are written for all b bodies as

$$\begin{bmatrix} \boldsymbol{M} \\ \mathbf{P} \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} \boldsymbol{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{0} \end{bmatrix} \tag{11.33}$$

where

$$\boldsymbol{M} = \begin{bmatrix} \mathbf{N}_1 & & & & \\ & 2\mathbf{J}_1'\mathbf{L}_1 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathbf{N}_b & \\ & & & & 2\mathbf{J}_b'\mathbf{L}_b \end{bmatrix} \tag{11.34}$$

$$\boldsymbol{b} = \begin{bmatrix} \boldsymbol{0} \\ \mathbf{L}_1\mathbf{H}_1\dot{\mathbf{p}}_1 \\ \vdots \\ \boldsymbol{0} \\ \mathbf{L}_b\mathbf{H}_b\dot{\mathbf{p}}_b \end{bmatrix} \tag{11.35}$$

$$\mathbf{g} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{n}_1' \\ \vdots \\ \mathbf{f}_b \\ \mathbf{n}_b' \end{bmatrix} \tag{11.36}$$

Formulation III. Equations 11.8 and 11.18 are written for all b bodies to obtain a set of equations identical to Eq. 8.40:

$$\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g} \tag{11.37}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{N}_1 & & & & \\ & \mathbf{J}_1' & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & \mathbf{N}_b & \\ & & & & \mathbf{J}_b' \end{bmatrix} \tag{11.38}$$

$$\dot{\mathbf{h}} = \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\omega}}'_1 \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \dot{\boldsymbol{\omega}}'_b \end{bmatrix} \tag{11.39}$$

$$\mathbf{b} = \begin{bmatrix} \mathit{0} \\ \tilde{\boldsymbol{\omega}}'_1\mathbf{J}'_1\boldsymbol{\omega}'_i \\ \vdots \\ \mathit{0} \\ \tilde{\boldsymbol{\omega}}'_b\mathbf{J}'_b\boldsymbol{\omega}'_b \end{bmatrix} \tag{11.40}$$

11.4.2 Constrained Bodies

For a system of b constrained bodies with the m independent constraint equations Eq. 11.19, three different formulations are obtained. The second-time derivative of the constraint equations, i.e.,

$$\boldsymbol{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \gamma \tag{11.41}$$

is appended to the equations of motion. The total number of equations becomes equal to the total number of accelerations and Lagrange multipliers.

Formulation I. Equations 11.21, 11.22, and 11.15 are written for all b bodies and then Eq. 11.41 is appended to them to obtain $8 \times b + m$ equations, as follows:

$$\begin{bmatrix} \mathbf{M}^* & \mathbf{P}^T & \mathbf{B}^T \\ \mathbf{P} & \mathbf{0} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\sigma} \\ -\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{b}^* \\ \mathbf{c} \\ \mathit{0} \end{bmatrix} = \begin{bmatrix} \mathbf{g}^* \\ \mathit{0} \\ \gamma \end{bmatrix} \tag{11.42}$$

where

$$\begin{aligned} \mathbf{B} &= \boldsymbol{\Phi}_{\mathbf{q}} \\ &= [\boldsymbol{\Phi}_{\mathbf{r}_1}, \boldsymbol{\Phi}_{\mathbf{p}_1}, \ldots, \boldsymbol{\Phi}_{\mathbf{r}_b}, \boldsymbol{\Phi}_{\mathbf{p}_b}] \end{aligned} \tag{11.43}$$

and

$$\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_m]^T \tag{11.44}$$

Note that the square matrix at the left in Eq. 11.42 is symmetric.

Formulation II. Equations 11.21, 11.23, and 11.15 are written for all b bodies and then Eq. 11.41 is appended to them to obtain $7 \times b + m$ equations, as follows:

$$\begin{bmatrix} M & B^T \\ \mathbf{P} & \mathbf{0} \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} b \\ \mathbf{c} \\ \mathit{0} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathit{0} \\ \gamma \end{bmatrix} \tag{11.45}$$

where

$$B = [\Phi_{r_1}, \tfrac{1}{2}\Phi_{p_1}L_1^T, \ldots, \Phi_{r_b}, \tfrac{1}{2}\Phi_{p_b}L_b^T] \tag{11.46}$$

Note that the square matrix at the left in Eq. 11.45 is nonsymmetric.

Formulation III. In the last formulation, the acceleration equation as given by Eq. 11.41 is written in terms of the angular acceleration of each body, $\dot{\boldsymbol{\omega}}_i'$, instead of $\ddot{\mathbf{p}}_i$. This conversion is performed first by writing Eq. 11.41 as

$$[\Phi_{r_1}, \Phi_{p_1}, \ldots, \Phi_{r_b}, \Phi_{p_b}] \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \ddot{\mathbf{p}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \ddot{\mathbf{p}}_b \end{bmatrix} = \gamma \tag{11.47}$$

From the identity $\ddot{\mathbf{p}}_i = \frac{1}{2}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \frac{1}{4}\omega^2\mathbf{p}_i$, a typical term $\Phi_{p_i}\ddot{\mathbf{p}}_i$ in Eq. 11.47 can be written as

$$\begin{aligned} \Phi_{p_i}\ddot{\mathbf{p}}_i &= \Phi_{p_i}(\tfrac{1}{2}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \tfrac{1}{4}\omega_i^2\mathbf{p}_i) \\ &= \tfrac{1}{2}\Phi_{p_i}\mathbf{L}_i^T\dot{\boldsymbol{\omega}}_i' - \tfrac{1}{4}\omega_i^2\Phi_{p_i}\mathbf{p}_i \end{aligned}$$

Hence, Eq. 11.47 is rewritten as

$$[\Phi_{r_1}, \tfrac{1}{2}\Phi_{p_1}\mathbf{L}_1^T, \ldots, \Phi_{r_b}, \tfrac{1}{2}\Phi_{p_b}\mathbf{L}_b^T] \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\omega}}_1' \\ \vdots \\ \ddot{\mathbf{r}}_b \\ \dot{\boldsymbol{\omega}}_b' \end{bmatrix} = \gamma^{\#} \tag{11.48}$$

where the terms $-\frac{1}{4}\omega_i^2\Phi_{p_i}\mathbf{p}_i$, $i = 1, \ldots, b$, have been moved to the right side of the equation. A detailed explanation of this new form of the Jacobian matrix is given in the next section. Appending Eq. 11.48 to Eqs. 11.21 and 11.24 for all b bodies yields

$$\begin{bmatrix} \mathbf{M} & B^T \\ B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{h}} \\ -\boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \gamma^{\#} \end{bmatrix} \tag{11.49}$$

Note that the square matrix at the left in this equation is symmetric.

11.5 CONVERSION OF KINEMATIC EQUATIONS

Although the Euler parameters are ideal for representing the angular orientation of a body in space, they yield too many equations when their time derivatives are used explicitly in the equations of motion, as was shown in Eq. 11.16. Equation 11.18 shows that only three rotational equations are needed if $\dot{\boldsymbol{\omega}}_i'$ is used instead of $\ddot{\mathbf{p}}_i$. For a constrained body, the equations of motion given in Eq. 11.24 contain only three equations and also take advantage of the Euler parameters (the constraint equations and hence the Jacobian matrix are described in terms of Euler parameters). This advantage becomes apparent when Eqs. 11.42 and 11.49 are compared. Equation 11.49 contains $2 \times b$ fewer equations than Eq. 11.42.

In Eq. 11.49, the kinematic constraints are kept in terms of Euler parameters, as follows:

$$\begin{aligned}\mathbf{\Phi} &\equiv \mathbf{\Phi}(\mathbf{q}) \\ &= \mathbf{\Phi}(\mathbf{r}_1, \mathbf{p}_1, \ldots, \mathbf{r}_b, \mathbf{p}_b) = \mathbf{0}\end{aligned} \tag{11.50}$$

The velocity equations are written as

$$\begin{aligned}\dot{\mathbf{\Phi}} &\equiv \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} \\ &= [\mathbf{\Phi}_{\mathbf{r}_1}, \mathbf{\Phi}_{\mathbf{p}_1}, \ldots, \mathbf{\Phi}_{\mathbf{r}_b}, \mathbf{\Phi}_{\mathbf{p}_b}] \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\mathbf{p}}_1 \\ \vdots \\ \dot{\mathbf{r}}_b \\ \dot{\mathbf{p}}_b \end{bmatrix} \\ &= \left[\mathbf{\Phi}_{\mathbf{r}_1}, \tfrac{1}{2}\mathbf{\Phi}_{\mathbf{p}_1}\mathbf{L}_1^T, \ldots, \mathbf{\Phi}_{\mathbf{r}_b}, \tfrac{1}{2}\mathbf{\Phi}_{\mathbf{p}_b}\mathbf{L}_b^T\right] \begin{bmatrix} \dot{\mathbf{r}}_1 \\ \boldsymbol{\omega}_1' \\ \vdots \\ \dot{\mathbf{r}}_b \\ \boldsymbol{\omega}_b' \end{bmatrix} = \mathbf{0}\end{aligned} \tag{11.51}$$

The modified Jacobian matrix of Eq. 11.51 is the same as that of Eq. 11.48. The modified Jacobian matrix and the modified vector $\boldsymbol{\gamma}^\#$ can be obtained in explicit forms for the constraint equations of Chap. 7.

Example 11.1

The modified velocity and acceleration equations and hence the modified Jacobian matrix and vector $\boldsymbol{\gamma}^\#$ for the constraint equation of Eq. 7.3 are derived here. The velocity equation is

$$\begin{aligned}\dot{\Phi}^{(n1,1)} &\equiv \mathbf{s}_i^T\dot{\mathbf{s}}_j + \mathbf{s}_j^T\dot{\mathbf{s}}_i \\ &= \mathbf{s}_i^T\dot{\mathbf{A}}_j\mathbf{s}_j' + \mathbf{s}_j^T\dot{\mathbf{A}}_i\mathbf{s}_i' \\ &= \mathbf{s}_i^T\mathbf{A}_j\tilde{\boldsymbol{\omega}}_j'\mathbf{s}_j' + \mathbf{s}_j^T\mathbf{A}_i\tilde{\boldsymbol{\omega}}_i'\mathbf{s}_i' \\ &= -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j' - \mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i' \\ &= [-\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i', -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'] \begin{bmatrix} \boldsymbol{\omega}_i' \\ \boldsymbol{\omega}_j' \end{bmatrix} = 0\end{aligned} \tag{1}$$

The entries of the modified transformation matrix could have been found directly from Table 7.2:

$$\begin{aligned}\tfrac{1}{2}\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}\mathbf{L}_i^T &= \tfrac{1}{2}(2\mathbf{s}_j^T\mathbf{G}_i\tilde{\mathbf{s}}_i')\mathbf{L}_i^T \\ &= -\mathbf{s}_j^T\mathbf{G}_i(-\tilde{\mathbf{s}}_i'\mathbf{L}_i + \mathbf{s}_i'\mathbf{p}_i^T)^T \\ &= -\mathbf{s}_j^T\mathbf{G}_i\mathbf{L}_i^T\tilde{\mathbf{s}}_i' \\ &= -\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\end{aligned} \tag{2}$$

This result agrees with the coefficient of $\boldsymbol{\omega}_i'$ in Eq. 1. The time derivative of Eq. 1 yields

$$\ddot{\boldsymbol{\Phi}}^{(n1,1)} \equiv [-\mathbf{s}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i', -\mathbf{s}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'] \begin{bmatrix} \dot{\boldsymbol{\omega}}_i' \\ \dot{\boldsymbol{\omega}}_j' \end{bmatrix} + (\dot{\mathbf{s}}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i' - \mathbf{s}_j^T\dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i')\boldsymbol{\omega}_i' + (-\dot{\mathbf{s}}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j' - \mathbf{s}_i^T\dot{\mathbf{A}}_j\tilde{\mathbf{s}}_j')\boldsymbol{\omega}_j' = 0 \quad (3)$$

The last two terms of Eq. 3 can be simplified to obtain

$$\begin{aligned} \boldsymbol{\gamma}^{\#(n1,1)} &\equiv \dot{\mathbf{s}}_j^T\mathbf{A}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i' + \mathbf{s}_j^T\dot{\mathbf{A}}_i\tilde{\mathbf{s}}_i'\boldsymbol{\omega}_i' + \dot{\mathbf{s}}_i^T\mathbf{A}_j\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j' + \mathbf{s}_i^T\dot{\mathbf{A}}_j^T\tilde{\mathbf{s}}_j'\boldsymbol{\omega}_j \\ &= \dot{\mathbf{s}}_j^T\tilde{\mathbf{s}}_i\boldsymbol{\omega}_j + \mathbf{s}_j^T\tilde{\boldsymbol{\omega}}_i\tilde{\mathbf{s}}_i\boldsymbol{\omega}_i + \dot{\mathbf{s}}_i^T\tilde{\mathbf{s}}_j\boldsymbol{\omega}_j + \mathbf{s}_i^T\tilde{\boldsymbol{\omega}}_j\tilde{\mathbf{s}}_j\boldsymbol{\omega}_j \\ &= -2\dot{\mathbf{s}}_j^T\dot{\mathbf{s}}_i - \mathbf{s}_j^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i - \mathbf{s}_i^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_j \\ &= -2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i \end{aligned} \quad (4)$$

This example illustrates how the modified Jacobian matrix and vector $\boldsymbol{\gamma}^{\#}$ can be calculated. Table 11.1 shows the components of the Jacobian matrix and vector $\boldsymbol{\gamma}^{\#}$ for some of the most common constraints. This table provides sufficient information to assemble in the form of Eq. 11.49 the complete set of equations of motion for mechanical systems with the more commonly used constraints. Numerical methods for solving these equations are discussed in Chap. 13.

TABLE 11.1 Components in the Expansion of the Most Common Constraints[13]

$\boldsymbol{\Phi}$	$\boldsymbol{\Phi}_{\mathbf{r}_i}^{(m)}$	$\frac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_i}^{(m)}\mathbf{L}_i^T$	$\boldsymbol{\Phi}_{\mathbf{r}_j}^{(m)}$	$\frac{1}{2}\boldsymbol{\Phi}_{\mathbf{p}_j}^{(m)}\mathbf{L}_j^T$	$\boldsymbol{\gamma}^{\#}$
$\boldsymbol{\Phi}^{(n1,1)}$	$\mathbf{0}^T$	$-\mathbf{s}_j^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{0}^T$	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j + \dot{\mathbf{s}}_i^T\tilde{\boldsymbol{\omega}}_i\mathbf{s}_j + \dot{\mathbf{s}}_j^T\tilde{\boldsymbol{\omega}}_j\mathbf{s}_i$
$\boldsymbol{\Phi}^{(n2,1)}$	$-\mathbf{s}_i^T$	$-(\mathbf{d} + \mathbf{s}_i^B)^T\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{s}_i^T$	$-\mathbf{s}_i^T\tilde{\mathbf{s}}_j^B\mathbf{A}_j$	$-2\dot{\mathbf{d}}^T\dot{\mathbf{s}}_i - \mathbf{d}^T\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i + \mathbf{s}_i^T(\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^B - \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^B)$
$\boldsymbol{\Phi}^{(p1,2)}$	$\mathbf{0}$	$\tilde{\mathbf{s}}_j\tilde{\mathbf{s}}_i\mathbf{A}_i$	$\mathbf{0}$	$-\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_j\mathbf{A}_j$	$-2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{s}}_j + \tilde{\mathbf{s}}_j\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i - \tilde{\mathbf{s}}_i\tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j$
$\boldsymbol{\Phi}^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$(\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_i^B + \tilde{\mathbf{d}}\tilde{\mathbf{s}}_i)\mathbf{A}_i$	$\tilde{\mathbf{s}}_i$	$-\tilde{\mathbf{s}}_i\tilde{\mathbf{s}}_j^B\mathbf{A}_j$	$-2\tilde{\dot{\mathbf{s}}}_i\dot{\mathbf{d}} + \tilde{\mathbf{s}}_i(\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^B - \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^B) + \tilde{\mathbf{d}}\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i$
$\boldsymbol{\Phi}^{(s,3)}$	$\mathbf{I}$	$-\tilde{\mathbf{s}}_i^P\mathbf{A}_i$	$-\mathbf{I}$	$\tilde{\mathbf{s}}_j^P\mathbf{A}_j$	$-\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^P + \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^P$
$\boldsymbol{\Phi}^{(s-s,1)}$	$-2\mathbf{d}^T$	$2\mathbf{d}^T\tilde{\mathbf{s}}_i^P\mathbf{A}_i$	$2\mathbf{d}^T$	$-2\mathbf{d}^T\tilde{\mathbf{s}}_j^P\mathbf{A}_j$	$-2\dot{\mathbf{d}}^T\dot{\mathbf{d}} + 2\mathbf{d}^T(\tilde{\boldsymbol{\omega}}_i\dot{\mathbf{s}}_i^P - \tilde{\boldsymbol{\omega}}_j\dot{\mathbf{s}}_j^P)$

PROBLEMS

11.1 Show that matrix $\mathbf{J}_i^*$ is singular.

11.2 Express matrix $\mathbf{J}_i^*$ in terms of $\mathbf{J}_i$ and $\mathbf{G}_i$.

11.3 From the rotational equations of motion given in Eq. 11.6,

(a) Determine the inverse of the coefficient matrix $\begin{bmatrix} \mathbf{J}^* & \mathbf{p} \\ \mathbf{p}^T & 0 \end{bmatrix}_i$

(b) Solve Eq. 11.16 for vector $[\ddot{\mathbf{p}}^T, \sigma]_i^T$

(c) Show that σ_i obtained in part (b) is the same as that shown in Eq. 11.13.

11.4 Solve Eq. 11.17 for $\ddot{\mathbf{p}}_i$.

11.5 The kinetic energy of a body is defined as

$$T_i = \tfrac{1}{2}\dot{\mathbf{r}}_i^T \mathbf{N}_i \dot{\mathbf{r}}_i + \tfrac{1}{2}\boldsymbol{\omega}_i'^T \mathbf{J}_i' \boldsymbol{\omega}_i'$$

Express the kinetic energy in terms of:

(a) $\boldsymbol{\omega}_i$ *and* $\mathbf{J}_i$

(b) $\dot{\mathbf{p}}_i$ and $\mathbf{J}_i^*$

11.6 The angular orientation of a body is given as $\mathbf{p}_i = [0.5, 0.7, -0.5, 0.1]^T$. Point P on this body has the local coordinates $\mathbf{s}_i'^P = [1, -1, 2]^T$. A force $\vec{f}$ acts on this body and has local components $\mathbf{f}_i' = [3, -2, -1]^T$. Find the components of the moment of this force in the following forms:;

(a) $\mathbf{n}_i'$

(b) $\mathbf{n}_i$

(c) $\mathbf{n}_i^*$

11.7 Two bodies are connected to each other by a spherical joint as shown in Fig. P.11.7. In addition to the gravitational force, two external moments, $\vec{n}_1$ and $\vec{n}_2$, and one external force, $\vec{f}_1$, act on the bodies, where $\vec{f}_1$ is parallel to the y axis.

(a) Write the equations of motion for the bodies in terms of angular accelerations.

(b) Show the elements of the vector of forces in terms of the applied loads.

(c) From the equations of motion, show the components of the reaction forces acting at P on body i and body j.

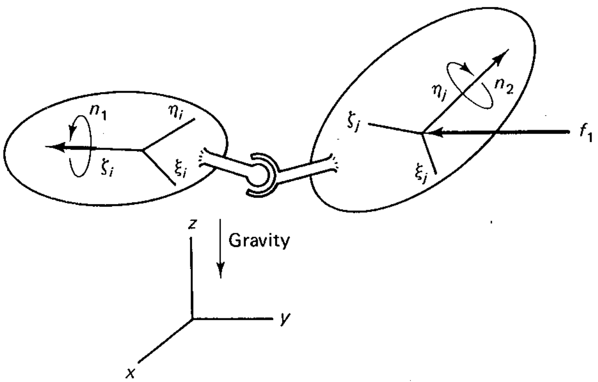

Figure P. 11.7

11.8 Verify the entries of the modified Jacobian matrix and vector $\gamma^{\#}$ listed in Table 11.1.