9

Planar Dynamics

In this chapter, the equations of motion for both unconstrained and constrained mechanical systems undergoing planar motion are developed in a form adequate for computer programming. Suitable equations are formulated for a variety of forces commonly encountered in mechanical systems, such as gravity and the forces of springs and dampers. The kinematic constraint equations of Chap. 4 are applied to complete the equations of motion.

9.1 EQUATIONS OF MOTION

Translational and rotational equations of motion for an unconstrained body are written from Eqs. 8.51 and 8.52, as follows:

$$m_i \ddot{x}_i = f_{(x)_i}$$
$$m_i \ddot{y}_i = f_{(y)_i}$$
$$j_{\zeta\zeta_i} \ddot{\phi}_i = n_i$$

or

$$\begin{bmatrix} m & & \\ & m & \\ & & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \tag{9.1}$$

where for notational simplicity the *polar moment of inertia* $j_{\zeta\zeta}$ of a body is denoted by μ. Equation 9.1 may also be expressed as

$$\mathbf{M}_i \ddot{\mathbf{q}}_i = \mathbf{g}_i \tag{9.2}$$

where

$$\mathbf{M}_i = \text{diag}\,[m, m, \mu]_i$$
$$\mathbf{q}_i = [x, y, \phi]_i^T$$
$$\mathbf{g}_i = [f_{(x)}, f_{(y)}, n]_i^T$$

A comparison of Eqs. 9.2 and 8.35 reveals that in planar motion $\mathbf{b}_i = \mathbf{0}$ and $\dot{\mathbf{h}}_i = \ddot{\mathbf{q}}_i$ or $\mathbf{h}_i = \dot{\mathbf{q}}_i$. The ambiguity that was mentioned in Sec. 8.4.2 between $\mathbf{h}_i$ and $\dot{\mathbf{q}}_i$, for general motion of a body, does not exist when planar motion is considered. In planar motion, the rotational velocity vector $\boldsymbol{\omega}_i' \equiv [0, 0, \dot{\phi}]_i^T$ is the time derivative of a rotational coordinate vector $[0, 0, \phi]_i^T$.

For a system of b unconstrained bodies, Eq. 9.2 is repeated b times as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{g} \tag{9.3}$$

where

$$\mathbf{M} = \text{diag}\,[\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_b]$$
$$\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_b^T]^T$$
$$\mathbf{g} = [\mathbf{g}_1^T, \mathbf{g}_2^T, \ldots, \mathbf{g}_b^T]^T$$

The system mass matrix $\mathbf{M}$ is a $3b \times 3b$ constant diagonal matrix, and vectors $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$, and $\mathbf{g}$ are $3b$-vectors.

For a system of b constrained bodies, the equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{g} + \mathbf{g}^{(c)} \tag{9.4}$$

where $\mathbf{g}^{(c)}$ is the vector of constraint reaction forces. Since Eq. 9.4, and hence $\mathbf{g}^{(c)}$, is described in the same coordinate system as $\mathbf{q}$, then from Eq. 8.50 it is found that

$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathrm{q}}^T\boldsymbol{\lambda} \tag{9.5}$$

where $\mathbf{\Phi} = \mathbf{0}$ represents m independent constraint equations. Substitution of Eq. 9.5 in Eq. 9.4 yields

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_{\mathrm{q}}^T\boldsymbol{\lambda} = \mathbf{g} \tag{9.6}$$

Equation 9.6 and the constraint equations

$$\mathbf{\Phi} = \mathbf{0} \tag{9.7}$$

together represent the equations of motion for a system of constrained bodies.

In kinematic analysis, the number of degrees of freedom of a system must be equal to the number of driver constraint equations. This means that m kinematic constraint equations and k driver equations provide n equations in n unknowns and so will yield a unique solution. However, in dynamic analysis, in general, there are no driver equations to be specified. Since $n > m$, there are more unknowns in the constraint equations of Eq. 9.7 than there are equations, and so there is no unique solution to these equations. In dynamic analysis, a unique solution is obtained when the constraint equations are considered simultaneously with the differential equations of motion, and a proper set of initial conditions is specified. These algebraic-differential equations are solved by numerical methods and will be discussed in Chap. 13.

9.2 VECTOR OF FORCES

Vector $\mathbf{g}$ in Eq. 9.6 contains the vectors of forces acting on all the bodies in the system; i.e.,

$$\mathbf{g} = [\mathbf{g}_1^T, \mathbf{g}_2^T, \ldots, \mathbf{g}_b^T]^T \tag{9.8}$$

To construct vector $\mathbf{g}$, the vector of force for each body must be determined. For a typical body i, the vector of force $\mathbf{g}_i$ contains all forces and moments acting on that body:

$$\mathbf{g}_i = [f_{(x)}, f_{(y)}, n]_i^T$$

where $f_{(x)_i}, f_{(y)_i}$, and n_i are the sums of all forces in the x and y directions and the sum of all moments, respectively. In Secs. 9.2.1 to 9.2.7, a variety of external and internal forces that commonly appear in mechanical systems are discussed, and their contributions to the elements of $\mathbf{g}_i$, and hence $\mathbf{g}$, are shown.

9.2.1 Gravitational Force

Figure 9.1 shows a body acted upon by a gravitational field in the negative y direction. The choice of the negative y direction as the direction of gravity is totally arbitrary. However, in this text the gravitational field will be considered to be acting in this direction in planar motion unless indicated otherwise.

If w_i is the weight of body i (mass of body i times the gravitational constant), then the contribution of this force to the vector of force of body i is

$$\mathbf{g}_i^{(\text{gravity})} = [0, -w, 0]_i^T \tag{9.9a}$$

9.2.2 Single Force or Moment

Consider a single force $\vec{f}_i$ acting with known direction at point P_i on body i as shown in Fig. 9.2(a). This force has components $f_{(x)_i}$ *and* $f_{(y)_i}$. If the local coordinates of P_i are known as $\mathbf{s}_i'^P = [\xi^P, \eta^P]_i^T$, then $\mathbf{s}_i^P = \mathbf{A}_i\mathbf{s}_i'^P$. The moment of $\vec{f}$ about the origin of the body is

$$\begin{aligned} n_i &= (\tilde{\mathbf{s}}_i^P \mathbf{f}_i)_{(z)} \\ &= s_{(y)_i}^P f_{(x)_i} + s_{(x)_i}^P f_{(y)_i} \\ &= -(\xi_i^P \sin \phi_i + \eta_i^P \cos \phi_i) f_{(x)_i} + (\xi_i^P \cos \phi_i - \eta_i^P \sin \phi_i) f_{(y)_i} \end{aligned} \tag{9.10}$$

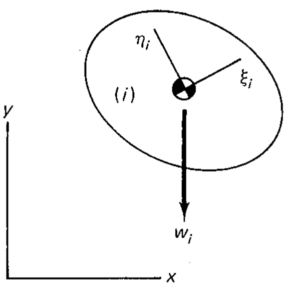


Figure 9.1 Gravitational field acting on a body.

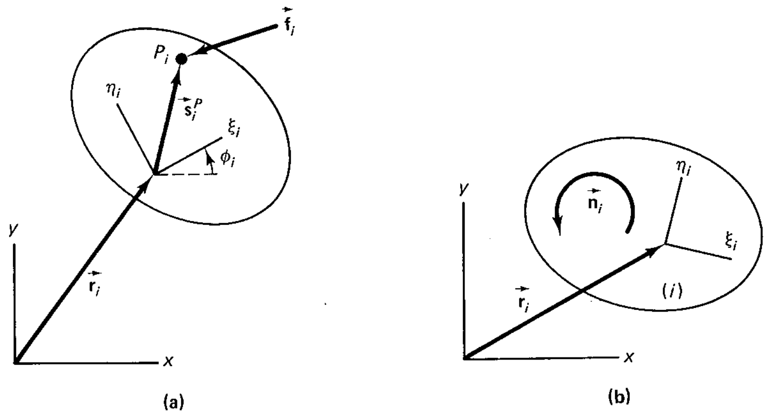


Figure 9.2 A body acted upon by a constant (a) force, and (b) moment.

The contribution of this force to the vector of forces of body i is

$$\mathbf{g}_i^{(\text{single } f)} = [f_{(x)}, f_{(y)}, n]_i^T \tag{9.11}$$

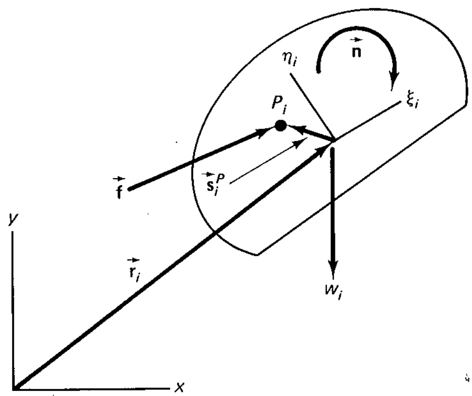
When a pure moment with magnitude n_i acts on body i as shown in Fig. 9.2(b), its contribution to the vector of forces of body i is

$$\mathbf{g}_i^{(\text{single } n)} = [0, 0, n]_i^T \tag{9.12}$$

Equations 9.11 and 9.12 are valid for either constant or time-dependent forces or moments.

Example 9.1

Body i, with a mass of 2, is acted upon by gravity, a constant force, and a pure moment, as shown in the illustration. The constant force has the components $\mathbf{f} = [1.2, 0.5]^T$, and the magnitude of the pure moment is 0.6. Determine the vector of force for this body if $\mathbf{s}_i'^P = [-0.2, 0.3]^T$, $\phi_i = 30°$, and $\mathbf{r}_i = [2.1, 1.6]^T$.



Solution The weight of the body is $w_i = 2 \times 9.81 = 19.62$. The moment of the force is found from Eq. 9.10 to be $n_i = -0.35$. Therefore, the vector of forces for this body is

$$\mathbf{g}_i = \begin{bmatrix} 1.2 \\ 0.5 - 19.62 \\ -0.35 - 0.6 \end{bmatrix} = \begin{bmatrix} 1.2 \\ -19.12 \\ -0.95 \end{bmatrix}$$

9.2.3 Translational Actuators

Actuators provide a constant or a time-dependent pair of forces acting on two bodies without imposing any kinematic constraints. The forces making up the pair have a common line of action but are in opposite directions. As shown in Fig. 9.3(a) an actuator acts between bodies i and j at the attachment points P_i and P_j. The equivalent representation for this system is shown in Fig. 9.3(b) or (c), depending on the direction of the forces.

The sign convention for the pair of forces can be defined as positive when the forces pull on the bodies and negative when the forces push on the bodies. If the actuator force is denoted by $f^{(a)}$, $f^{(a)} > 0$ constitutes a pull and $f^{(a)} < 0$ constitutes a push. In order to find the forces being applied to bodies i and j, i.e., $\vec{f}_i^{(a)}$ and $\vec{f}_j^{(a)}$, a unit vector on the line of action of the actuator must be defined.

A vector $\vec{l}$ connecting points P_i and P_j, as shown in Fig. 9.4, is defined as

$$\mathbf{l} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P \tag{9.13}$$

The magnitude of this vector is

$$l = (\mathbf{l}^T\mathbf{l})^{1/2} \tag{9.14}$$

A unit vector $\vec{u}$ is defined as

$$\mathbf{u} = \frac{\mathbf{l}}{l} \tag{9.15}$$

The unit vector $\vec{u}$ has the same direction as $\vec{f}_i^{(a)}$ in the case of a pull and $\vec{f}_j^{(a)}$ in the case of a push. Therefore,

$$\mathbf{f}_i^{(a)} = f^{(a)}\mathbf{u} \tag{9.16}$$

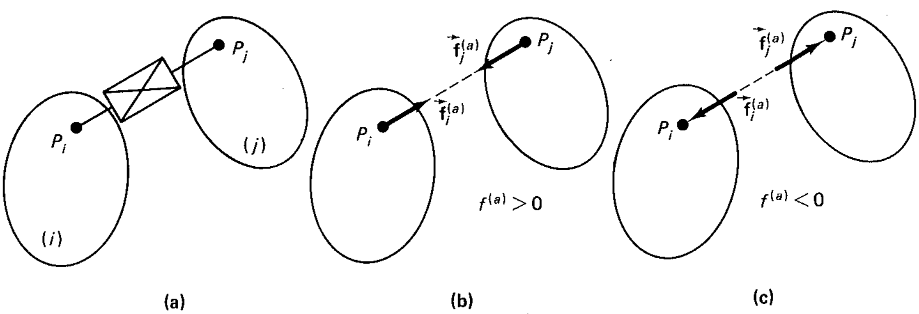


Figure 9.3 (a) An actuator acting between two bodies, and the equivalent representation; (b) pull; (c) push.

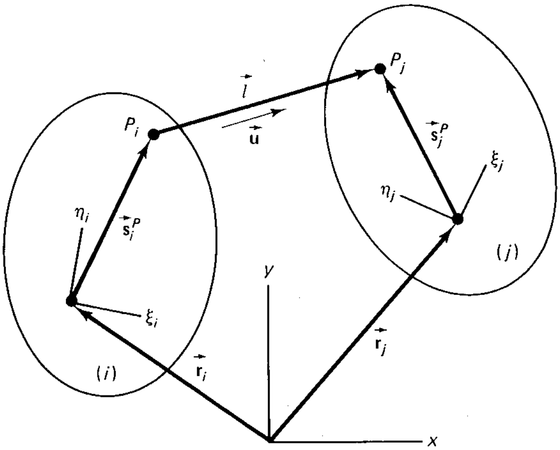


Figure 9.4 Defining a unit vector along the axis of the actuator forces.

and

$$\mathbf{f}_j^{(a)} = -f^{(a)}\mathbf{u} \tag{9.17}$$

It is clear that since $f^{(a)}$ can be either positive or negative, the sign convention in Eqs. 9.16 and 9.17 is automatically satisfied. The contribution of $\mathbf{f}_i^{(a)}$ (or $\mathbf{f}_j^{(a)}$) to the vector of forces $\mathbf{g}_i$(or $\mathbf{g}_j$) can be found from Eqs. 9.10 and 9.11.

9.2.4 Translational Springs

Translational (point-to-point) springs are the most commonly used force elements in mechanical systems. Figure 9.5(a) shows a spring attached between points P_i and P_j on bodies i and j. The force of this spring can be found as

$$f^{(s)} = k(l - l^0) \tag{9.18}$$

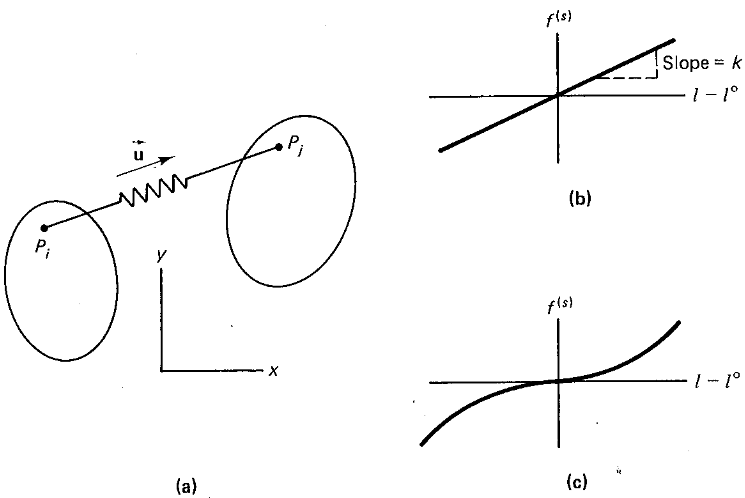


Figure 9.5 (a) A translational spring between two bodies with (b) linear characteristics or (c) nonlinear characteristics.

where k is the spring stiffness, l is the deformed length, and l^0 is the undeformed length of the spring. The deformed length of the spring is found from Eq. 9.14.

The sign convention for the spring force is similar to that of the actuator force—positive in *tension* (pull) and negative in compression (push). The forces of the spring acting on bodies i and j are

$$\mathbf{f}_i^{(s)} = f^{(s)}\mathbf{u} \tag{9.19}$$

and

$$\mathbf{f}_j^{(s)} = -f^{(s)}\mathbf{u} \tag{9.20}$$

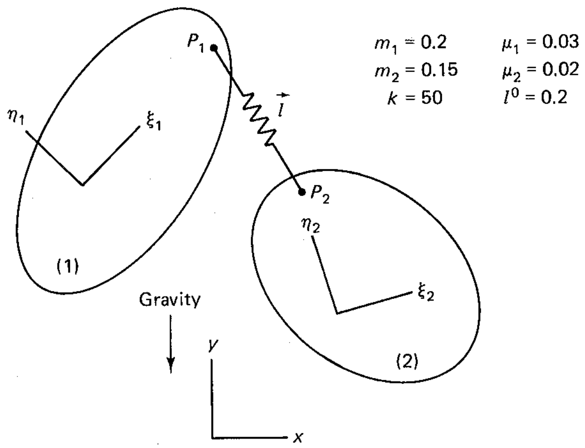
where $\mathbf{u}$ is a unit vector defined along $\boldsymbol{l}$ (Eq. 9.15). Equations 9.19 and 9.20 are valid in tension and compression—if $l > l^0$ (for tension), $f^{(s)}$ is positive; and if $l < l^0$ (for compression), $f^{(s)}$ is negative.

The contributions of $\mathbf{f}_i^{(s)}$ (or $\mathbf{f}_j^{(s)}$) to $\mathbf{g}_i$ (or $\mathbf{g}_j$) are found from Eqs. 9.10 and 9.11.

In Eq. 9.18 a linear characteristic is assumed for the spring (Fig. 9.5(b)). However, the spring may have nonlinear force-deformation characteristics, e.g., the curve shown in Fig. 9.5(c). In this case, the force-deformation curve can be used directly instead of Eq. 9.18. If the force-deformation data are available in discretized form, the linear or cubic spline function technique (Sec. 4.2.4) can be employed to compute $f^{(s)}$ for a deformation $l - l^0$.

Example 9.2

Two bodies are connected by a translational spring, where $\mathbf{s}_1'^P = [0.15, 0]^T$ and $\mathbf{s}_2'^P = [0, 0.1]^T$ (see the illustration). Write the equations of motion when $\mathbf{q}_1 = [-0.1, 0.2, 0.785]^T$ and $\mathbf{q}_2 = [0.1, 0.1, 0.262]^T$, and then calculate the accelerations.



Solution From Eq. 9.13, vector $\boldsymbol{l}$ is found to be equal to $[0.068, -0.109]^T$, and hence $l = 0.129$. The unit vector along $\boldsymbol{l}$ is $\mathbf{u} = [0.528, -0.849]^T$. The spring force is $f^{(s)} = 50(0.129 - 0.2) = -3.558$. From Eqs. 9.19 and 9.20 it is found that $\mathbf{f}_1^{(s)} = [-1.878, 3.022]^T$ and $\mathbf{f}_2^{(s)} = [1.878, -3.022]^T$. Equation 9.10 yields $n_1^{(s)} = 0.520$ and $n_2^{(s)} = -0.103$. The vectors of forces for bodies 1 and 2 contain the contribution from the spring and from gravity and are found to be

$\mathbf{g}_1 = [-1.878, 1.060, 0.520]^T$ and $\mathbf{g}_2 = [1.878, -4.494, -0.103]^T$. The equations of motion are written from Eq. 9.3 as

$$\begin{bmatrix} 0.2 & & & & & \\ & 0.2 & & & & \\ & & 0.03 & & & \\ & & & 0.15 & & \\ & & & & 0.15 & \\ & & & & & 0.02 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \end{bmatrix} = \begin{bmatrix} -1.878 \\ 1.060 \\ 0.520 \\ 1.878 \\ -4.494 \\ -0.103 \end{bmatrix}$$

The accelerations are found easily to be

$$\ddot{\mathbf{q}} = [-9.389, 5.302, 17.326, 12.518, -29.959, -5.154]^T$$

9.2.5 Translational Dampers

A translational (point-to-point) damper between two bodies i and j is shown in Fig. 9.6. The damping force can be found to be

$$f^{(d)} = d\dot{l} \tag{9.21}$$

where d is the damping coefficient and $\dot{l}$ is the time rate of change of the damper length. $\dot{l}$ is found by taking the time derivative of Eq. 9.14:

$$\dot{l} = \frac{\mathbf{l}^T\dot{\mathbf{l}}}{l} \tag{9.22}$$

where $\dot{\mathbf{l}}$, in turn, is found from Eq. 9.13:

$$\dot{\mathbf{l}} = \dot{\mathbf{r}}_j + \dot{\phi}_j\mathbf{B}_j\mathbf{s}_j'^P - \dot{\mathbf{r}}_i - \dot{\phi}_i\mathbf{B}_i\mathbf{s}_i'^P \tag{9.23}$$

and where

$$\mathbf{B}_k = \begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix}_k \qquad k = i, j$$

The sign convention for the damping force is defined as positive for $\dot{l} > 0$ and negative for $\dot{l} < 0$. Since a damper opposes the relative motion of two bodies, when the two bodies move away from each other (when $\dot{l} > 0$), the forces of the damper exhibit a pull, and when the bodies move toward each other (when $\dot{l} < 0$), the forces of the damper exhibit a push.

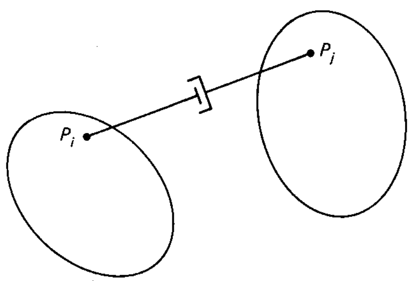


Figure 9.6 A translational damper between two bodies.

By defining a unit vector $\vec{u}$ along $\vec{l}$, we express the forces $\mathbf{f}_i^{(d)}$ and $\mathbf{f}_j^{(d)}$ as

$$\mathbf{f}_i^{(d)} = f^{(d)}\mathbf{u} \tag{9.24}$$

and

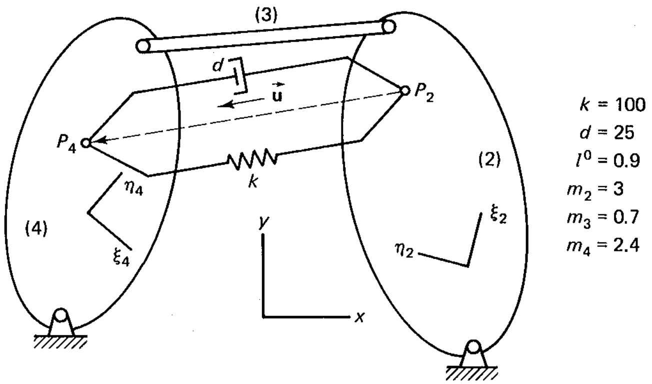
$$\mathbf{f}_j^{(d)} = -f^{(d)}\mathbf{u} \tag{9.25}$$

Equations 9.24 and 9.25 are valid for both pull and push cases.

As was true of deformation in the case of springs, the relationship between force and deformation rate for a damper can be linear or nonlinear. Equation 9.21 assumes a linear characteristic for the damper. However, if the damper characteristic is nonlinear, a curve or a table of data describing the relationship between force and deformation rate can be used instead of Eq. 9.21.

Example 9.3

A spring-damper element is connected between bodies 2 and 4 of a four-bar linkage, as shown in the illustration. The attachment points are $\mathbf{s}_2'^P = [0.3, 0.2]^T$ and $\mathbf{s}_4'^P = [-0.1, 0.1]^T$. If at a particular instant the vectors of coordinates and velocities are $\mathbf{q}_2 = [0.4, 0.1, 1.3]^T$, $\mathbf{q}_4 = [-0.35, 0.2, 5.6]^T$, $\dot{\mathbf{q}}_2 = [0.8, -0.6, -0.3]^T$, and $\dot{\mathbf{q}}_4 = [-0.5, 0.45, -0.1]^T$, determine the vector of forces for the three moving bodies.



Solution From Eqs. 9.13 and 9.23 it can be found that $\boldsymbol{l} = [-0.652, -0.102,]^T$ and $\dot{\boldsymbol{l}} = [-1.389, 1.018]^T$, which yield $l = 0.660$ and $\dot{l} = 1.215$. A unit vector along $\boldsymbol{l}$ is $\mathbf{u} = [-0.988, -0.154]^T$. Equations 9.18 and 9.21 yield the spring and damper forces $f^{(s)} = 100(0.660 - 0.9) = -24.012$ and $f^{(d)} = 25 \times 1.215 = 30.373$. Since the spring and the damper have the same point of application on each body, their total force can be used as $f^{(s+d)} = -24.012 + 30.373 = 6.361$. The components of this force acting on the two bodies are

$$\mathbf{f}_2^{(s+d)} = \begin{bmatrix} -6.285 \\ -0.982 \end{bmatrix} \qquad \mathbf{f}_4^{(s+d)} = \begin{bmatrix} 6.285 \\ 0.982 \end{bmatrix}$$

Equation 9.10 can be used to determine that the moments of these forces are $n_2^{(s+d)} = 2.263$ and $n_4^{(s+d)} = -0.898$.

The weights of the bodies are $w_2 = 29.430$, $w_3 = 6.867$, and $w_4 = 23.544$. The vectors of forces for the bodies are

$$\mathbf{g}_2 = \begin{bmatrix} -6.285 \\ -30.412 \\ 2.263 \end{bmatrix} \quad \mathbf{g}_3 = \begin{bmatrix} 0.0 \\ -6.867 \\ 0.0 \end{bmatrix} \quad \mathbf{g}_4 = \begin{bmatrix} 6.285 \\ -22.562 \\ -0.898 \end{bmatrix}$$

9.2.6 Rotational Springs

A rotational (torsional) spring acting between two bodies i and j is shown in Fig. 9.7(a). The two bodies are also assumed to be connected by a revolute joint whose axis is the same as the spring axis. A rotational spring applies pure moments on the bodies, equal in magnitude and opposite in direction.

The moment is found as

$$n^{(r\text{-}s)} = k(\theta - \theta^0) \tag{9.26}$$

where k is the spring stiffness, θ is the deformed angle of the spring, and θ^0 is the undeformed angle, as shown in Fig. 9.7(b). Vectors $\vec{s}_i$ and $\vec{s}_j$ are assumed to be attached to the spring in order to define the spring angle.

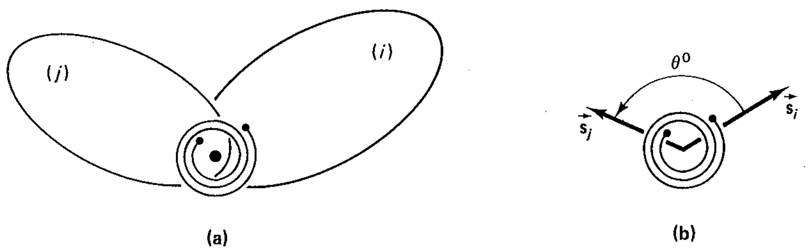


Figure 9.7 (a) A rotational spring acting between two bodies. (b) Free (undeformed) state of the spring.

When $\theta > \theta^0$, the moment of the spring acts on body i in the positive rotational direction and on body j in the negative rotational direction, as shown in Fig. 9.8(a). When $\theta < \theta^0$ the situation is reversed, as shown in Fig. 9.8(b). Therefore,

$$n_i^{(r\text{-}s)} = n^{(r\text{-}s)} \tag{9.27}$$

and

$$n_j^{(r\text{-}s)} = -n^{(r\text{-}s)} \tag{9.28}$$

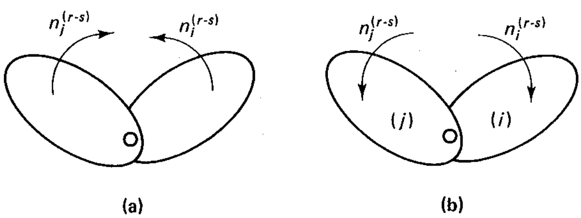


Figure 9.8 The moments of a rotational spring acting on the bodies, (a) for $\theta > \theta^0$ and (b) for $\theta < \theta^0$.

9.2.7 Rotational Dampers

The rotational element shown in Fig. 9.7 may also contain a damper, in addition to the spring. For a rotational damper the moment is found as

$$n^{(r\text{-}d)} = d\dot{\theta} \tag{9.29}$$

where d is the damping coefficient and

$$\dot{\theta} = \dot{\phi}_j - \dot{\phi}_i \tag{9.30}$$

is the time rate of change of the element angle.

When $\dot{\theta} > 0$, the moment of the damper acts on body i in the positive rotational direction and on body j in the negative rotational direction. When $\dot{\theta} < 0$, the situation is reversed. Therefore,

$$n_i^{(r\text{-}d)} = n^{(r\text{-}d)} \tag{9.31}$$

and

$$n_j^{(r\text{-}d)} = -n^{(r\text{-}d)} \tag{9.32}$$

9.3 CONSTRAINT REACTION FORCES

The joint reaction forces can be expressed in terms of the Jacobian matrix of the constraint equations and a vector of Lagrange multipliers, as shown in Eq. 8.50, as

$$\mathbf{g}^{(c)} = \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} \tag{9.33}$$

This equation is studied for several commonly used constraints in Secs. 9.3.1 to 9.3.3.

9.3.1 Revolute Joint

Consider two bodies i and j connected by a revolute joint, as shown in Fig. 9.9(a). The kinematic constraint equations for this joint are given by Eq. 4.7. The equations of motion for bodies i and j are

$$\mathbf{M}_i\ddot{\mathbf{q}}_i - \mathbf{\Phi}_{\mathbf{q}_i}^T\boldsymbol{\lambda} = \mathbf{g}_i \tag{a}$$

and

$$\mathbf{M}_j\ddot{\mathbf{q}}_j - \mathbf{\Phi}_{\mathbf{q}_j}^T\boldsymbol{\lambda} = \mathbf{g}_j \tag{b}$$

Using the entries of the Jacobian matrix for a revolute joint from Table 4.2, we can write Eq. a in the expanded form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_i - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -(y_i^P - y_i) & (x_i^P - x_i) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ n \end{bmatrix}_i \tag{c}$$

Since there are two algebraic equations in the constraint equations for a revolute joint, vector $\boldsymbol{\lambda}$ is correspondingly a 2-vector. Equation c can be written as the set of equations

$$m_i\ddot{x}_i = f_{(x)_i} + \lambda_1 \tag{9.34}$$

$$m_i\ddot{y}_i = f_{(y)_i} + \lambda_2 \tag{9.35}$$

$$\mu_i\ddot{\phi}_i = n_i - (y_i^P - y_i)\lambda_1 + (x_i^P - x_i)\lambda_2 \tag{9.36}$$

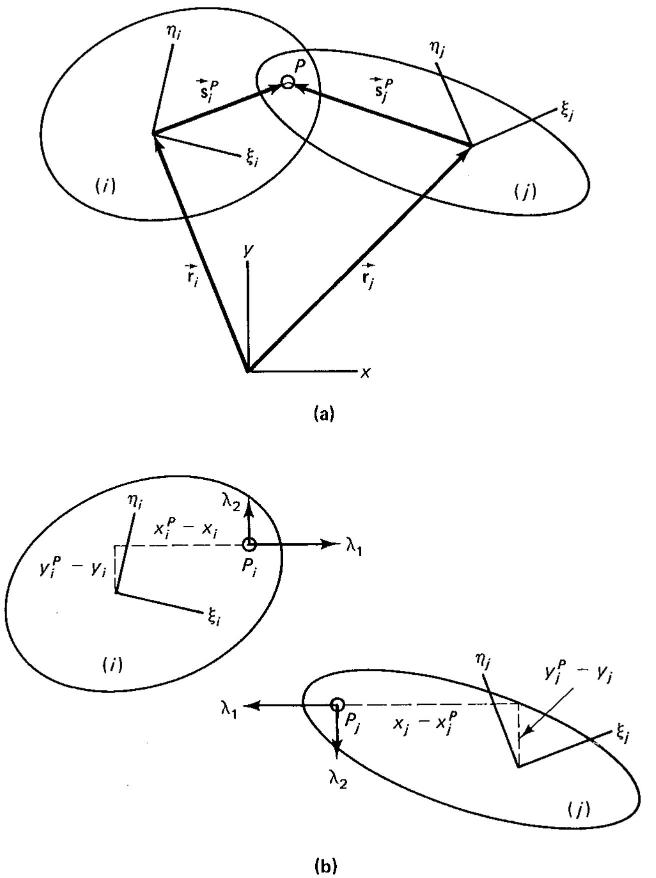


Figure 9.9 (a) Two bodies connected by a revolute joint; (b) free-body diagrams for the bodies.

A free-body diagram for body i is shown in Fig. 9.9(b). Equation 9.34 indicates that besides $f_{(x)_i}$, another force, λ_1, acts in the x direction on body i. Similarly, from Eq. 9.35 it is deduced that a force λ_2 acts in the y direction on the same body. However, in order for Eq. 9.36 to be satisfied, forces λ_1 and λ_2 must act at point P_i. The moment arm of λ_1 is $y_i^P - y_i$, and hence a moment $(y_i^P - y_i)\lambda_1$ acts in the negative rotational direction. The moment arm of λ_2 is $x_i^P - x_i$, and so a moment $(x_i^P - x_i)\lambda_2$ acts in the positive rotational direction.

Equations of motion for body j, in the same form as Eq. c are written as follows:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mu \end{bmatrix}_j \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\phi} \end{bmatrix}_j - \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ (y_j^P - y_j) & -(x_j^P - x_j) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{(x)} \\ \mathbf{f}_{(y)} \\ n \end{bmatrix}_j \qquad (d)$$

or

$$m_j\ddot{x}_j = f_{(x)_j} - \lambda_1 \qquad (9.37)$$

$$m_j\ddot{y}_j = f_{(y)_j} - \lambda_2 \qquad (9.38)$$

$$\mu_j\ddot{\phi}_j = n_j + (y_j^P - y_j)\lambda_1 - (x_j^P - x_j)\lambda_2 \qquad (9.39)$$

It is shown in Fig. 9.9(b) that λ_1 and λ_2 are two forces acting at point P_j in the negative x and y directions, respectively. The moment arm for λ_1 is $y_j^P - y_j$ which yields a positive moment $(y_j^P - y_j)\lambda_1$, and the moment arm for λ_2 is $x_j - x_j^P$, which yields a moment $(x_j - x_j^P)\lambda_2$ or $-(x_j^P - x_j)\lambda_2$.

The multipliers λ_1 and λ_2 can be positive or negative quantities. In any case, the reaction forces acting at the revolute joint on the connecting bodies are always equal in magnitude and opposite in direction.

Example 9.4

Consider a system of two bodies connected by a revolute joint as shown in Fig. 9.9(a). The external forces acting on the system are gravity, a constant force of 10 N acting on body i in the negative x direction, and a constant force of 10 N acting on body j in the positive x direction. Calculate the joint reaction forces at the instant for which

$$\mathbf{q}_i = [1.58, 1.59, 0.6]^T, \qquad \mathbf{q}_j = [3.4, 1.96, 0.2]^T$$
$$\dot{\mathbf{q}}_i = [1.1, 0.2, -0.02]^T, \qquad \dot{\mathbf{q}}_j = [1.14, 0.24, 0.03]^T$$

The constant quantities for this system are: $m_i = 1.2, m_j = 2, \mu_i = 2.5, \mu_j = 4$, $\mathbf{s}_i'^P = [0.9, 0.7]^T$, and $\mathbf{s}_j'^P = [-1.3, 1]^T$.

Solution The constraint equations for this revolute joint are

$$x_i + 0.9 \cos \phi_i - 0.7 \sin \phi_i - x_j + 1.3 \cos \phi_j + \sin \phi_j = 0$$
$$y_i + 0.9 \sin \phi_i + 0.7 \cos \phi_i - y_j + 1.3 \sin \phi_j - \cos \phi_j = 0$$

The Jacobian matrix for these constraints is

$$\mathbf{\Phi}_\mathbf{q} = \begin{bmatrix} 1 & 0 & -1.09 & -1 & 0 & 0.72 \\ 0 & 1 & 0.35 & 0 & -1 & 1.47 \end{bmatrix}$$

From Eqs. 9.34 through 9.36, the equations of motion for body i are

$$\begin{aligned} 1.2\ddot{x}_i - \lambda_1 &= -10 \\ 1.2\ddot{y}_i - \lambda_2 &= -11.77 \\ 2.5\ddot{\phi}_i + 1.09\lambda_1 - 0.35\lambda_2 &= 0 \end{aligned} \tag{1}$$

Similarly, Eqs. 9.37 through 9.39 provide equations of motion for body j:

$$\begin{aligned} 2\ddot{x}_j + \lambda_1 &= 10 \\ 2\ddot{y}_j + \lambda_2 &= -19.62 \\ 4\ddot{\phi}_j - 0.72\lambda_1 - 1.47\lambda_2 &= 0 \end{aligned} \tag{2}$$

Equations 1 and 2 are six equations in eight unknowns, and therefore two more equations are needed. These two additional equations are the kinematic acceleration equations. The second-time derivative of the constraint equations (refer to Table 4.3) can be used to obtain the acceleration equations for the revolute joint, as follows:

$$\begin{aligned} \ddot{x}_i - 1.09\ddot{\phi}_i - \ddot{x}_j + 0.72\ddot{\phi}_j &= 0 \\ \ddot{y}_i + 0.35\ddot{\phi}_i - \ddot{y}_j + 1.47\ddot{\phi}_j &= 0 \end{aligned} \tag{3}$$

The right side of the acceleration equations is approximately zero, i.e., $\boldsymbol{\gamma} = [0.0017, -0.0002]^T$. Equations 1 through 3 can be solved to find

$\ddot{\mathbf{q}}_i = [-2.571, -10.154, -3.061]^T$, $\ddot{\mathbf{q}}_j = [1.543, -9.604, 1.096]^T$, and $\boldsymbol{\lambda} = [6.915, -0.413]^T$. Hence, $\mathbf{f}_i^{(c)} = [6.915, -0.413]^T$ and $\mathbf{f}_j^{(c)} = [-6.915, 0.413]^T$.

9.3.2 Revolute-Revolute Joint

Consider two bodies i and j connected by a revolute joint as shown in Fig. 9.10(a). The equations of motion for bodies i and j, using the elements of the Jacobian matrix for a revolute-revolute joint from Table 4.2, are written as

$$m_i \ddot{x}_i = f_{(x)_i} + 2(x_i^P - x_j^P)\lambda_1 \tag{9.40}$$

$$m_i \ddot{y}_i = f_{(y)_i} + 2(y_i^P - y_j^P)\lambda_1 \tag{9.41}$$

$$\mu_i \ddot{\phi}_i = n_i - 2[(x_i^P - x_j^P)(y_i^P - y_i) - (y_i^P - y_j^P)(x_i^P - x_i)]\lambda_1 \tag{9.42}$$

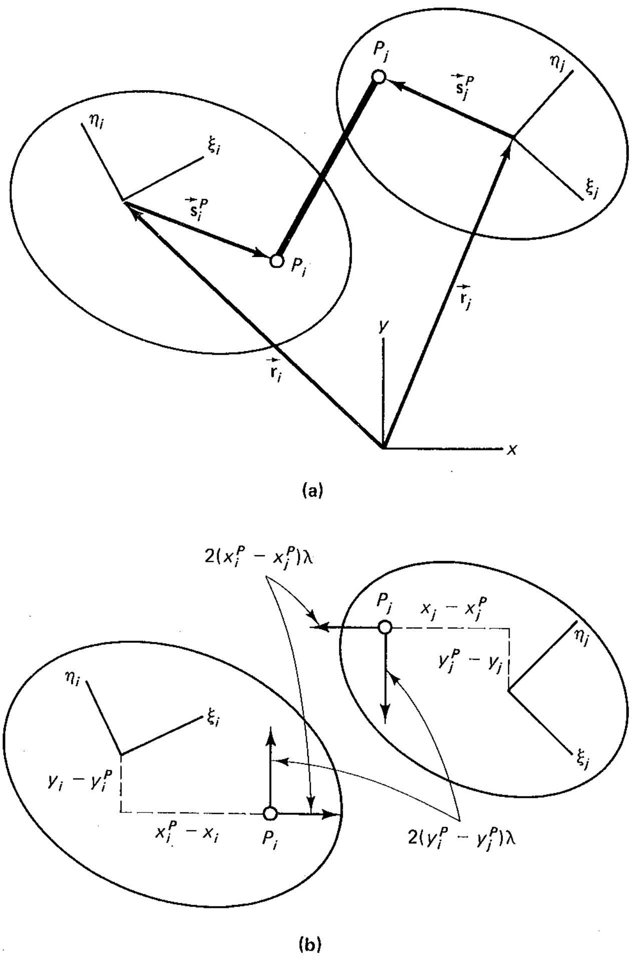


Figure 9.10 (a) Two bodies connected by a revolute-revolute joint. (b) Free-body diagrams for the bodies.

and

$$m_j \ddot{x}_j = f_{(x)_j} - 2(x_i^P - x_j^P)\lambda_1 \tag{9.43}$$

$$m_j \ddot{y}_j = f_{(y)_j} - 2(y_i^P - y_j^P)\lambda_1 \tag{9.44}$$

$$\mu_j \ddot{\phi}_j = n_j + 2[(x_i^P - x_j^P)(y_j^P - y_j) - (y_i^P - y_j^P)(x_i^P - x_j)]\lambda_1 \tag{9.45}$$

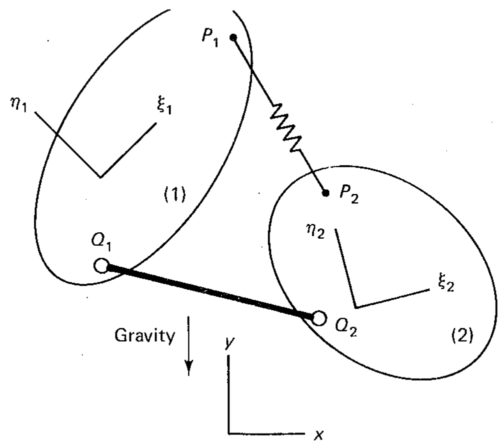
There is only one Lagrange multiplier in these equations; it corresponds to the one constraint equation describing the revolute-revolute joint.

From Eqs. 9.40 and 9.41 it is deduced that the terms $2(x_i^P - x_j^P)\lambda_1$ and $2(y_i^P - y_j^P)\lambda_1$ can be considered reaction forces acting on body i in the x and y directions, respectively. However, in order for Eq. 9.42 to be valid, these forces must act at point P_i. Figure 9.10(b) shows the components of the reaction force and the moment arms at point P_i. Similarly, Eqs. 9.43 and 9.44 show that the x and y components of the reaction force on body j are $-2(x_i^P - x_j^P)\lambda_1$ and $-2(y_i^P - y_j^P)\lambda_1$, and Eq. 9.45 indicates that these forces must act at point P_j.

The reaction forces at points P_i and P_j are equal in magnitude and opposite in direction. These forces act along the revolute-revolute joint axis, i.e., a line passing through points P_i and P_j.

Example 9.5

For the two-body system of Example 9.2, assume that a revolute-revolute joint with a length $l = 0.175$ is connected between points Q_1 and Q_2, where $\mathbf{s}_1'^Q = [-0.05, -0.05]^T$ and $\mathbf{s}_2'^Q = [-0.03, 0.0]^T$ (see the illustration). Write the equations of motion and calculate the reaction forces due to this added link if $\dot{\mathbf{q}}_1 = [0.0, 1.22, 0.0]^T$ and $\dot{\mathbf{q}}_2 = [-0.71, -2.06, 0.0]^T$.



Solution The global coordinates of Q_1 and Q_2 are found to be $\mathbf{r}_1^Q = [-0.1, 0.129]^T$ and $\mathbf{r}_2^Q = [0.071, 0.092]^T$. The equations of motion of Example 9.2 are modified as follows:

$$\begin{bmatrix} 0.2 & & & & & \\ & 0.2 & & & & \\ & & 0.03 & & & \\ & & & 0.15 & & \\ & & & & 0.15 & \\ & & & & & 0.02 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\phi}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{\phi}_2 \end{bmatrix} - \begin{bmatrix} -0.342 \\ 0.074 \\ 0.024 \\ 0.342 \\ -0.074 \\ -0.001 \end{bmatrix} \lambda_1 = \begin{bmatrix} -1.878 \\ 1.060 \\ 0.520 \\ 1.878 \\ -4.494 \\ -0.103 \end{bmatrix}$$

Since $\mathbf{\Phi}_\mathbf{q}^T$ is already available and γ can be found from Table 4.3, the kinematic acceleration equation for the revolute-revolute joint can now be found:

$$-0.342\ddot{x}_1 + 0.074\ddot{y}_1 + 0.024\ddot{\phi}_1 + 0.342\ddot{x}_2 - 0.074\ddot{y}_2 - 0.001\ddot{\phi}_2 = -22.525$$

There are seven equations in seven unknowns that are solved to find

$$\lambda_1 = -22.828$$

$$\ddot{\mathbf{q}} = [29.646, -3.146, -0.929, -39.528, -18.698, -4.008]^T$$

The reactions forces at Q_1 and Q_2 are $\mathbf{f}_1^{(c)} = [7.807, -1.689]^T$ and $\mathbf{f}_2^{(c)} = [-7.807, 1.689]^T$.

9.3.3 Translational Joint

If a translational joint is considered between bodies i and j as shown in Fig. 9.11(a), the equations of motion for body i can be written as

$$m_i\ddot{x}_i = f_{(x)_i} + (y_i^P - y_i^Q)\lambda_1 \tag{9.46}$$

$$m_i\ddot{y}_i = f_{(y)_i} - (x_i^P - x_i^Q)\lambda_1 \tag{9.47}$$

$$\mu_i\ddot{\phi}_i = n_i - [(x_j^P - x_i)(x_i^P - x_i^Q) + (y_j^P - y_i)(y_i^P - y_i^Q)]\lambda_1 + \lambda_2 \tag{9.48}$$

The free-body diagram for body i is shown in Fig. 9.11(b). In this diagram the force associated with λ_1 is the reaction force caused by the first constraint equation. It is a simple matter to show that this force, $\vec{k}_1$, is perpendicular to the line of translation. The contribution of the second constraint equation is a couple acting on body i. Note that λ_2 may be a positive or negative quantity.

In order to find a simpler physical description of the reaction force $\vec{k}_1$, one should not locate the points P_i, Q_i, and P_j arbitrarily on the line of translation. These points can be selected to coincide with the edges of the slider, as shown in Fig. 9.12(a). If P_j is allowed to slide with the slider, then the reaction force $\vec{k}_1$ always acts at the edge of the slider, as shown in Fig. 9.12(b).

9.4 SYSTEM OF PLANAR EQUATIONS OF MOTION

For an unconstrained mechanical system, the equations of motion are as given in Eq. 9.3:

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{g} \tag{9.49}$$

Vector $\mathbf{g}$ is, in general, a function of $\mathbf{q}$, $\dot{\mathbf{q}}$, and t. If, at an instant, $\mathbf{q}$ and $\dot{\mathbf{q}}$ are known, $\ddot{\mathbf{q}}$ can be found as follows:

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}\mathbf{g} \tag{9.50}$$

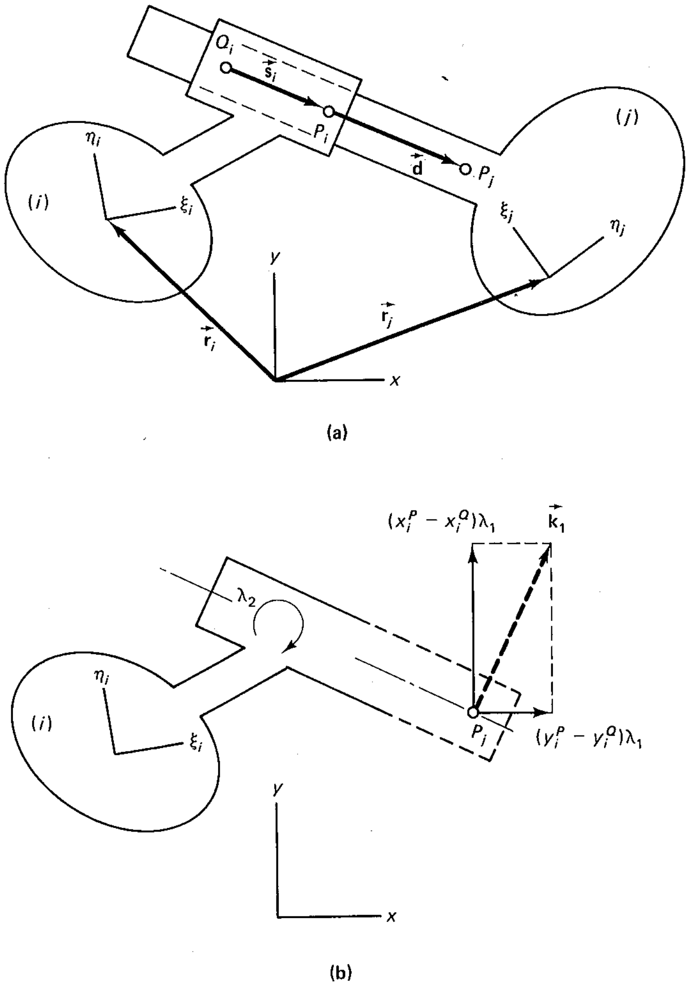


Figure 9.11 (a) Two bodies connected by a translational joint and (b) the reaction forces acting on body i associated with a translational joint.

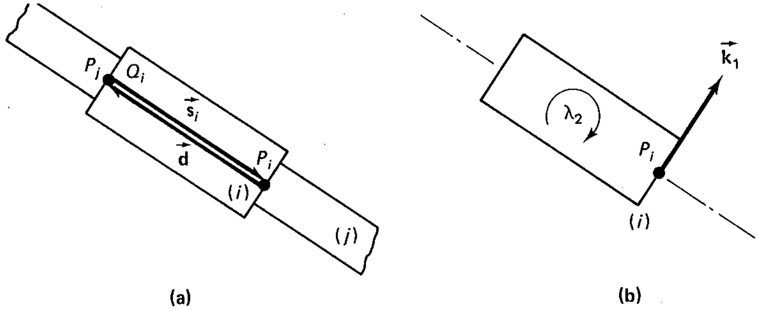


Figure 9.12 (a) A typical translational joint. (b) Forces acting on body i by the sliding body j.

For a well-posed problem, every body in the system must have nonzero mass and moment of inertia. Therefore, $\mathbf{M}$ is a diagonal nonsingular matrix, and $\mathbf{M}^{-1}$ can be calculated easily.

For a constrained mechanical system with m independent constraints

$$\mathbf{\Phi} = \mathbf{0} \tag{9.51}$$

the velocity and acceleration equations are

$$\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{0} \tag{9.52}$$

and

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} - \boldsymbol{\gamma} = \mathbf{0} \tag{9.53}$$

The equations of motion for this constrained system are as given in Eq. 9.6:

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{g} \tag{9.54}$$

Equation 9.53 can be appended to Eq. 9.54 and the result can be written as

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{bmatrix} \tag{9.55}$$

The Jacobian matrix $\mathbf{\Phi}_{\mathbf{q}}$ is a function of $\mathbf{q}$, and vectors $\mathbf{g}$ and $\boldsymbol{\gamma}$ are functions of $\mathbf{q}$, $\dot{\mathbf{q}}$, and t. Therefore, at any given instant, if $\mathbf{q}$ and $\dot{\mathbf{q}}$ are known, Eq. 9.55 provides $n + m$ linear algebraic equations in $n + m$ unknowns that can be solved for $\ddot{\mathbf{q}}$ and $\boldsymbol{\lambda}$. For constrained mechanical systems, Eqs. 9.51 through 9.54 must be considered together as the system equations of motion.

A FORTRAN program for solving the planar equations of motion is presented in Chap. 10. Numerical methods for solving ordinary differential equations (for unconstrained systems) and mixed algebraic-differential equations (for constrained systems) are discussed in detail in Chaps. 12 and 13.

9.5 STATIC FORCES

The subjects that are discussed in this section and in Secs. 9.6 and 9.7 are valid for both planar and spatial systems. However, because of the simplicity of illustrations for planar systems, these alone will be treated in this chapter.

A mechanical system becomes a structure (a nonmovable system) when the number of independent constraint equations is equal to the number of coordinates in the system. For example, the system shown in Fig. 9.13 contains 8 links and the ground, which yields $n = (8 + 1) \times 3 = 27$ coordinates. There are 12 revolute joints in the system, resulting in 24 algebraic equations, and 3 algebraic equations for the ground constraints, totaling $m = 24 + 3 = 27$. This yields $k = 27 - 27 = 0$ degree of freedom. In general, for a structure with n coordinates $\mathbf{q}$, n constraint equations can be written as

$$\mathbf{\Phi}(\mathbf{q}) = \mathbf{0}$$

These equations can be solved to find the coordinates $\mathbf{q}$. Since for a structure $\dot{\mathbf{q}} = \ddot{\mathbf{q}} = \mathbf{0}$, Eq. 9.4 yields

$$\mathbf{g}^{(c)} = -\mathbf{g} \tag{9.56}$$

This equation shows that the constraint reaction forces acting on each body of the system can be found directly from the vector of forces. In order to find the constraint reac-

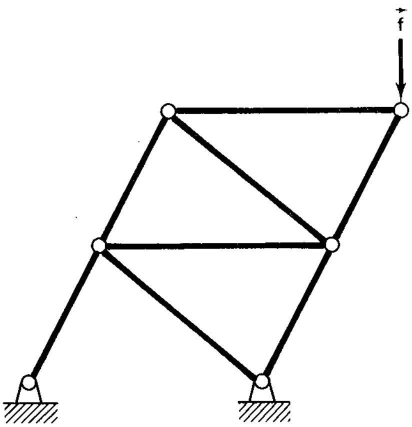


Figure 9.13 A planar truss subject to an external force.

tion forces at each joint, the Lagrange multipliers can be determined as

$$\boldsymbol{\lambda} = -(\boldsymbol{\Phi}_{\mathbf{q}}^T)^{-1}\mathbf{g} \tag{9.57}$$

The inverse of $\boldsymbol{\Phi}_{\mathbf{q}}^T$ exists, since it is assumed that the constraints are independent and, for a system with 0 degree of freedom, $\boldsymbol{\Phi}_{\mathbf{q}}$ is a square matrix. After the determination of $\boldsymbol{\lambda}$, a process similar to that of Secs. 9.3.1 to 9.3.3 can be employed to find the reaction forces at each point.

9.6 STATIC BALANCE FORCES

Consider the planar robot manipulator shown in Fig. 9.14. The motion of the robot is controlled by three electric motors (rotational actuators) acting about the axes of revolute joints A, B, and C. What moments must the motors apply on the bodies in order to keep the system in equilibrium, in the configuration shown? The moments (or forces, in other examples) are referred to as the static balance forces. If the number of unknown static balance forces is equal to the number of degrees of freedom, then the forces can be found by the following method.

The vector of forces is split into two vectors, as follows:

$$\mathbf{g} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)} \tag{9.58}$$

where $\mathbf{g}^{(k)}$ contains the known forces acting on the system and $\mathbf{g}^{(u)}$ is the vector of unknown forces, which can be the static balance forces. Hence, the equations of motion

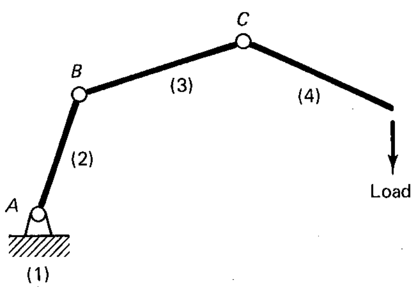


Figure 9.14 A planar robot manipulator.

for the system are written as

$$\mathbf{M}\ddot{\mathbf{q}} - \mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)}$$

or

$$-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)} \tag{a}$$

since in a static configuration $\ddot{\mathbf{q}} = \mathbf{0}$. The same mechanical system can be kept in equilibrium, in its given configuration, if the actuators with unknown forces (or moments) are replaced by artificial constraint equations, equal in number to the number of degrees of freedom. These artificial constraints are denoted here by k algebraic equations as

$$\mathbf{\Phi}^*(\mathbf{q}) = \mathbf{0} \tag{9.59}$$

For example, for the robot manipulator of Fig. 9.14, three artificial constraints are defined as

$$\Phi_1^* \equiv \phi_2 - \phi_1 - c_1 = 0$$
$$\Phi_2^* \equiv \phi_3 - \phi_2 - c_2 = 0$$
$$\Phi_3^* \equiv \phi_4 - \phi_3 - c_3 = 0$$

These constraints keep the relative angles between bodies constant.

If these equations are appended to the original m kinematic constraint equations $\mathbf{\Phi}(\mathbf{q}) = \mathbf{0}$, then the equations of motion become

$$-\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} - \mathbf{\Phi}_{\mathbf{q}}^{*T}\boldsymbol{\lambda}^* = \mathbf{g}^{(k)} \tag{9.60}$$

where $\ddot{\mathbf{q}} = \mathbf{0}$. Comparing Eq. a and Eq. 9.60 results in

$$\mathbf{g}^{(u)} = \mathbf{\Phi}_{\mathbf{q}}^{*T}\boldsymbol{\lambda}^* \tag{9.61}$$

Equation 9.60 represents n linear algebraic equations in n unknowns. The n unknowns are m multipliers $\boldsymbol{\lambda}$ and k multipliers $\boldsymbol{\lambda}^*$. If these equations are solved for $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^*$, then Eq. 9.61 yields the unknown static balance forces $\mathbf{g}^{(u)}$.

Example 9.6

The motion of the five-bar linkage in Fig. 9.15 is controlled by two actuators as shown in Fig. 9.15(a). The number of actuators is the same as the number of degrees of freedom. In order to find what forces applied by the actuators will keep the system in equilibrium, the actuators are replaced by two revolute-revolute joints, as shown in Fig. 9.15(b). The two artificial constraints for the revolute-

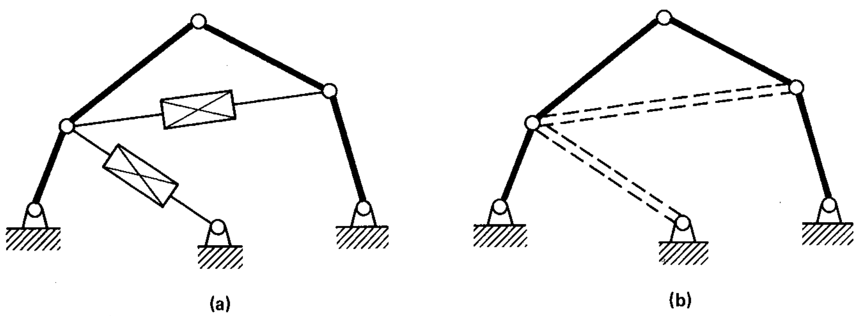


Figure 9.15 A five-bar mechanism.

revolute joints lower the number of degrees of freedom from 2 to 0. If the system of Fig. 9.15(b) is solved for the reaction forces along the revolute-revolute joints, then Eq. 9.61 will yield the desired actuator forces in the equivalent system of Fig. 9.15(a).

9.7 KINETOSTATIC ANALYSIS

If the forces acting on a mechanical system are known, then the equations of motion can be solved to obtain the motion of the system. This process is known as *forward dynamic analysis*. In some problems, a specified motion for a mechanical system is sought and the objective is to determine the forces that must act on the system to produce such a motion. This process is usually referred to as *inverse dynamic* or *kinetostatic analysis*.

As an example, consider the 3-degrees of freedom robot manipulators of Fig. 9.14. Assume that the end effector, point P, must move along a known path, such as the straight line shown in Fig. 9.16. The range of interest is from E to F, and it is further required that point P keep a constant velocity within this range. One additional requirement is that the angle of body 4 must remain unchanged with respect to the line EF. The objective is to find the torque that actuators A, B, and C must supply, as a function of time, in order to produce such motion.

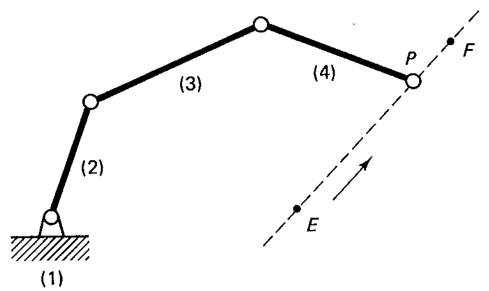


Figure 9.16 The end-effector of the robot must move along a specified path, keeping a specified orientation.

This problem can be solved by specifying k driving constraint equations, equal to the number of degrees of freedom, to describe the required motion—such as

$$\mathbf{\Phi}^*(\mathbf{q}, t) = \mathbf{0} \tag{9.62}$$

For example, the driving constraints for the robot manipulator are

$$\Phi_1^* \equiv x_4^P - a_1 - v_1 t = 0$$
$$\Phi_2^* \equiv y_4^P - a_2 - v_2 t = 0$$
$$\Phi_3^* \equiv \phi_4 - c_1 = 0$$

where a_1 and a_2 are the initial x and y coordinates of P at $t = 0$, v_1 and v_2 are the constant velocities of P along the x and y axes, and c_1 is the specified angle for body 4.

The k driving constraints of Eq. 9.62 are appended to m kinematic constraints to yield n constraints in n unknowns. This is a kinematics problem that can be solved by the method described in Sec. 3.2.2. The time t is varied from $t = 0$ to t^e in order to move point P from E to F. At every time step, position, velocity, and acceleration analyses are performed and the results, i.e., $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$, are saved in numerical form.

The equations of motion can be written as

$$\mathbf{M}\ddot{\mathbf{q}} - \boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{g}^{(k)} + \mathbf{g}^{(u)} \tag{9.63}$$

where $\mathbf{g}^{(k)}$ contains the known forces, such as gravity, and $\mathbf{g}^{(u)}$ contains the unknown forces of the actuators (the moments of the actuators A, B, and C in the robot example). Since $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$ are calculated kinematically, $\mathbf{M}\ddot{\mathbf{q}}$, $\boldsymbol{\Phi}_{\mathbf{q}}$, and $\mathbf{g}^{(k)}$ are known. Therefore,

$$\boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} + \mathbf{g}^{(u)} = \mathbf{M}\ddot{\mathbf{q}} - \mathbf{g}^{(k)} \tag{9.64}$$

can be solved for $\boldsymbol{\lambda}$ and $\mathbf{g}^{(u)}$. Equation 9.64 represents n equations in m unknowns $\boldsymbol{\lambda}$ and k unknowns embedded in $\mathbf{g}^{(u)}$. These equations can be solved at every time step from $t = 0$ to t^e, and the actuator forces can be found numerically as functions of time.

PROBLEMS

9.1 A force $\vec{f}$ acts at point P on body i as shown in Fig. P. 9.1. This force keeps a fixed angle α with vector $\vec{s}_i^P$. Find the component of this force and its corresponding moment for inclusion in vector $\mathbf{g}_i$.

9.2 Repeat Prob. 9.1 and assume that the force keeps a constant angle β with the global x axis as shown in Fig. P. 9.2.

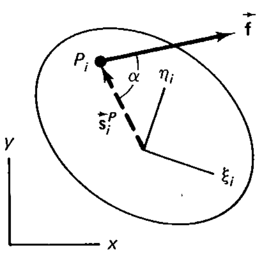


Figure P. 9.1

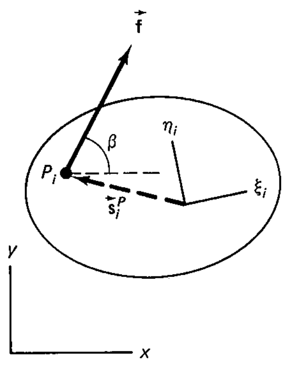


Figure P. 9.2

9.3 A multibody model of a vehicle is assembled in the configuration shown in Fig. P. 9.3(a), where the gravitational force is perpendicular to the road. If the vehicle is placed on a slope

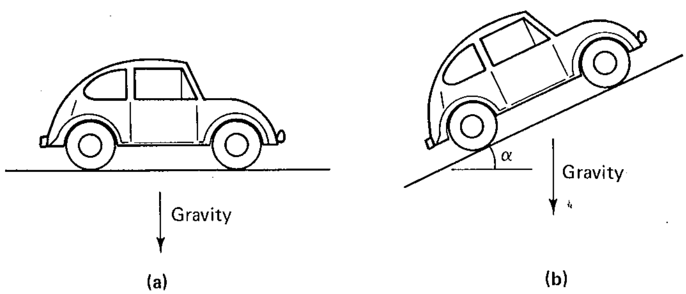


Figure P. 9.3

as shown in Fig. P. 9.3(b), the gravitational force makes an angle α with the normal to the road. Instead of changing the coordinate values from model (a) to model (b), devise a simple method to modify the vector of forces by rotating the direction of the gravitational force with respect to the global coordinate axes.

9.4 Derive the equations of motion for a body when the origin of the local coordinate system does not coincide with the body center of mass, as shown in Fig. P. 9.4.

9.5 For the single pendulum shown in Fig. P. 9.5, write the equations of motion in terms of Cartesian coordinates. Use the kinematic acceleration equations to eliminate the translational components of acceleration and Lagrange multipliers. The resultant equation should be a second-order differential equation in terms of $\ddot{\phi}_2$.

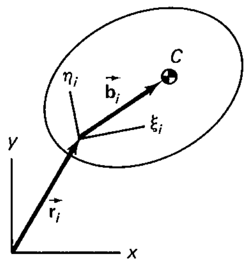


Figure P. 9.4

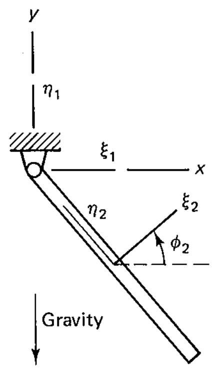


Figure P. 9.5

9.6 The rod shown in Fig. P. 9.6 is attached to the ground by a spring. Write the equations of motion for the rod. What are the initial conditions on the coordinates? Assume $m = 4$, $\mu = 3$, $k = 50$, and $l^0 = 1$.

9.7 Two unconstrained bodies are connected to each other and the ground by springs and dampers as shown in Fig. P. 9.7. Let $m_1 = 4$, $\mu_1 = 3$, $m_2 = 3$, $\mu_2 = 1$, $k = 40$, $l^0 = 1.2$, and $d = 12$.

(a) Define local and global coordinate systems.

(b) Determine the initial condition for the vector of coordinates.

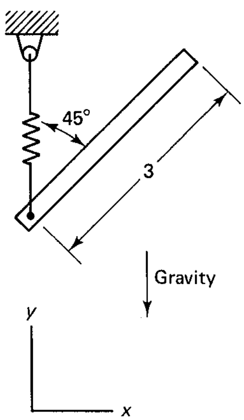


Figure P. 9.6

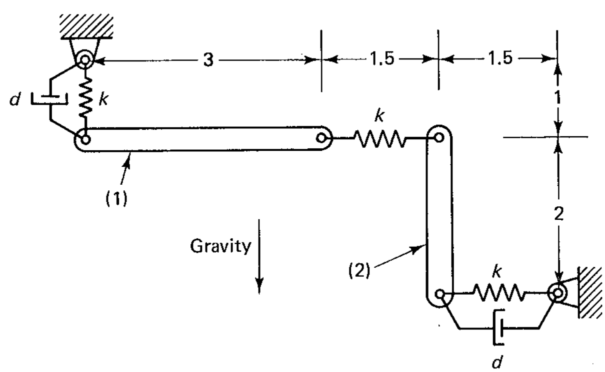


Figure P. 9.7

(c) If $\dot{x}_1 = -0.3$ and $\dot{\phi}_2 = 0.05$, determine the initial condition for the vector of velocities.

(d) Write the equations of motion for the system.

9.8 Two rods are connected to each other by a revolute joint as shown in Fig. P. 9.8. Let $m_1 = m_2 = 6$, $\mu_1 = \mu_2 = 12.5$, $k_1 = 20$, $l_1^0 = 5$, $k_2 = 30$, $l_2^0 = 4.5$, and $d_2 = 6$.

(a) Define local and global coordinate systems.

(b) Determine the initial condition for the vector of coordinates.

(c) Test the constraint equations for any violations. In case of violation, correct the initial conditions.

(d) If rod 1 has a positive rotational velocity of $\dot{\phi}_1 = 0.01$ rad/s, find a proper set of initial conditions for the vector of velocities consistent with the constraints.

(e) Write the equations of motion for the system.

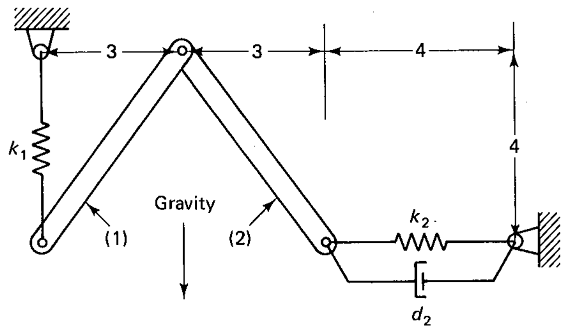


Figure P. 9.8

9.9 Two masses m_1 and m_2 go through a one-dimensional motion in the x direction as shown in Fig. P. 9.9. Assume that $m_1 = 1$, $m_2 = 2$, $k_1 = 10$, $k_2 = 15$, $l_1^0 = 1.25$, $l_2^0 = 1$, $d_1 = 5$, $d_2 = 6$, $a = 1$, and $b = 3$.

(a) Write the equations of motion for this system in terms of $\ddot{x}_1$ and $\ddot{x}_2$ (do not combine the two masses into a single mass).

(b) At the instant shown, $x_1 = 1.2$, $x_2 = 2.2$, and $\dot{x}_1 = \dot{x}_2 = 0.3$. Solve the equations of motion for the accelerations.

(c) Draw the free-body diagram for each mass and show all the forces in their proper directions.

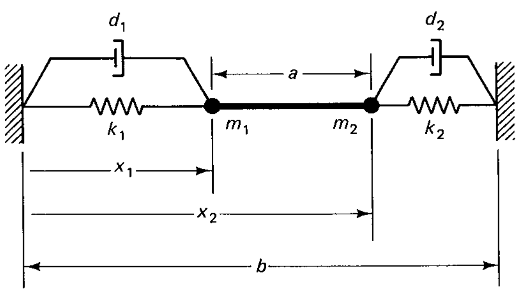


Figure P. 9.9

9.10 The radial deformation of an automobile wheel may be modeled by a translational spring-damper combination as long as the wheel is in contact with the ground. Knowing the radius, position, and velocity of the wheel, find:

(a) The coordinates of the contact point (center of the contact patch)
(b) The spring force
(c) The damper force
(d) The components of the resultant force acting on the wheel

Repeat this process for the three cases shown in Fig. P. 9.10. Assume that complete geometrical data for the road are available.

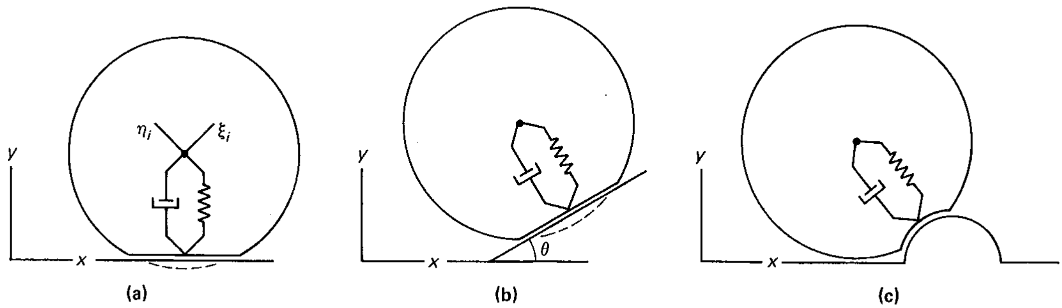


Figure P. 9.10

9.11 Repeat Prob. 9.10 for the case where the wheel and the ground are in contact at two points (patches) as shown in Fig. P. 9.11. The resultant force acting on the wheel can be found as the sum of forces from two spring-damper elements.

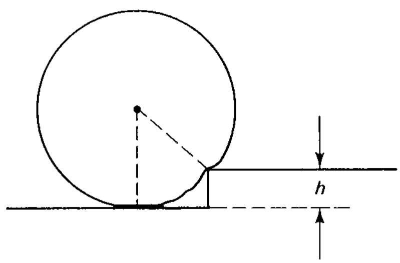


Figure P. 9.11

9.12 Deformation of the cantilever beam shown in Fig. P. 9.12(a) may be modeled by a rigid body, a revolute joint, and a rotational spring, as shown in Fig. P. 9.12(b). For a beam with

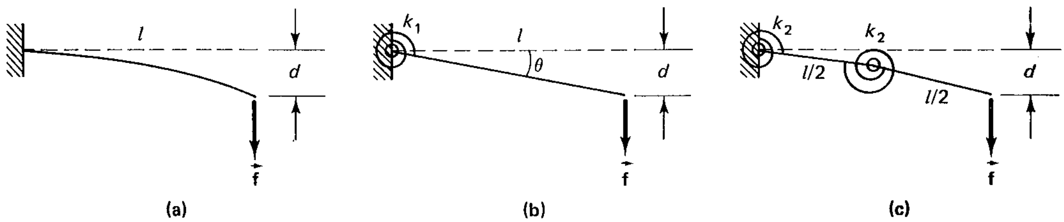


Figure P. 9.12

length l under an external load f, the free end yields a displacement d. The equivalent rigid-body model yields the same displacement if the spring stiffness k_1 is selected properly.

(a) If the beam is modeled by two rigid bodies, two revolute joints, and two rotational springs with stiffness k_2, as shown in Fig. P. 9.12(c), find an approximate formula for k_2 in terms of k_1 (for small deformations $d \ll l$).

(b) If the beam is modeled by n equal-length bodies, n revolute joints, and n rotational springs, find a formula for k_n in terms of k_1.

9.13 For two bodies connected by a revolute-translational joint, show that the reaction force between the bodies can be found from the term $\Phi_{\mathbf{q}}^T$ in the equations of motion. Show the forces on free-body diagrams of the two bodies.