8

Basic Concepts in Dynamics

The basic concepts and laws of dynamics are best introduced by beginning with particle dynamics. To derive the equations of motion for both unconstrained and constrained systems of bodies, only two of Newton's laws of motion for a single particle are needed as postulates.

8.1 DYNAMICS OF A PARTICLE

The simplest body arising in the study of motion is a *particle,* or point mass, defined here as a mass concentrated at a point. While mass in reality is generally distributed in space, the notion that a body has all its mass concentrated at a point is an adequate approximation for many purposes. Further, as will be seen shortly, the centroid of a complex distribution of mass behaves as a point mass. Thus, analysis of the behavior of a point mass leads to useful results, even for complex systems.

Newton's first law of motion for a particle relates the total *force* $\vec{f}$ acting on the particle, the *mass* $m^{(p)}$ of the particle, and the acceleration $\vec{a}$ of the particle, as follows (see Fig. 8.1):

$$\vec{f} = m^{(p)}\vec{a} \tag{8.1}$$

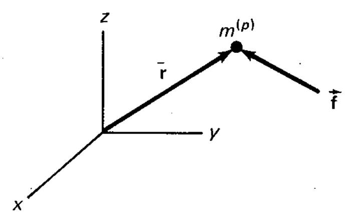

Figure 8.1 A particle moving in a global coordinate system.

where it is assumed that a consistent system of units is used. From this equation, a condition for particle *equilibrium* may be deduced. A particle remains at rest (in equilibrium) or in a state of constant velocity if and only if the total force $\vec{f}$ acting on the particle is $\vec{0}$.

Newton's second law of motion, which is the *law of action and reaction,* states that when two particles exert forces on each other, these interacting forces are equal in magnitude, opposite in sense, and directed along the straight line joining the particles.

The vector form of Newton's law of motion for a particle in Eq. 8.1 can be written in terms of the components of vectors $\vec{f}$ and $\vec{a}$. If we denote the force $\vec{f}$ as $\mathbf{f} = [f_{(x)}, f_{(y)}, f_{(z)}]^T$, then for a particle located in the xyz coordinate system with position vector $\vec{r}$, Eq. 8.1 becomes

$$\mathbf{f} = m^{(p)}\ddot{\mathbf{r}} \tag{8.2}$$

Throughout the preceding discussion of Newton's laws of motion, it is presumed that position, and hence acceleration, is measured in an inertial reference frame (a global xyz) coordinate system. Such a reference frame should technically be defined as a coordinate system fixed in the stars. For most engineering purposes, an adequate reference frame is an *earth-fixed reference system.* It is important, however, to note that for applications concerning space dynamics, or even in long-range trajectories, the rotation of the earth has a significant effect on the precision with which points can be located by means of Newton's equations of motion, and an earth-fixed reference system may be inadequate.

8.2 DYNAMICS OF A SYSTEM OF PARTICLES

The governing laws of the dynamics of individual particles are now extended to systems of interacting particles. The equations of motion for such systems can be written simply as the collection of equations of motion for all the particles taken individually. If the forces acting between particles are readily characterized, this method is practical and direct. The more common situation in mechanical system dynamics, however, involves constraints among systems of particles; thus the forces acting between particles are usually not so readily determined. For this reason we introduce the concept of the center of mass.

Consider the system of p particles shown schematically in Fig. 8.2. Particle i has mass $m_i^{(p)}$ and is located by a position vector $\vec{r}_i$ directed from the origin of an inertial reference frame to the particle. The forces acting on each particle include an externally applied force $\vec{f}_i$ and internal forces of interaction between particles $\vec{f}_{ij}$, $j \neq i$, where $\vec{f}_{ii} = \vec{0}$. The total force acting on the ith particle is the summation of external and internal forces. Thus, for body i, Newton's first law of motion (Eq. 8.2) becomes

$$m_i^{(p)}\ddot{\mathbf{r}}_i = \mathbf{f}_i + \sum_{j=1}^{p} \mathbf{f}_{ij}, \qquad i = 1, \ldots, p \tag{a}$$

This system of equations describes the motion of the system of p particles.

Since the forces of interaction between bodies in a system must satisfy Newton's law of action and reaction, the force on body i due to body j must be equal to the negative of the force acting on body j due to body i; i.e.,

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji} \qquad i,j = 1, \ldots, p \tag{b}$$

Note that since $\mathbf{f}_{ii} = \boldsymbol{0}$, Eq. b holds also for $i = j$.

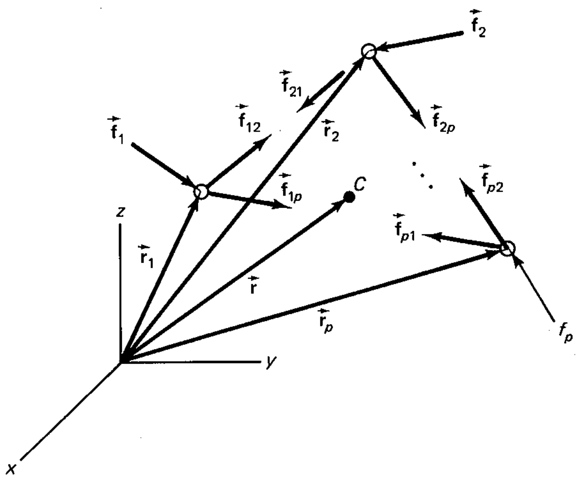

Figure 8.2 System of p particles.

If the external forces acting on each particle are known and the nature of the force acting between bodies i and j is known, the system of equations in Eq. a may be written explicitly. However, the force of interaction between particles will generally depend on the positions of the particles.

Since Eq. a must hold for each particle in the system, this system of equations can be summed to obtain

$$\sum_{i=1}^{p} m_i^{(p)}\ddot{\mathbf{r}}_i = \sum_{i=1}^{p} \mathbf{f}_i + \sum_{i=1}^{p} \sum_{j=1}^{p} \mathbf{f}_{ij} \tag{c}$$

which is valid for the entire system of particles. The double sum of Eq. c contains both $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$, and hence from Eq. b it is found that

$$\sum_{i=1}^{p} \sum_{j=1}^{p} \mathbf{f}_{ij} = \mathit{0} \tag{d}$$

Thus, Eq. c reduces to

$$\sum_{i=1}^{p} m_i^{(p)}\ddot{\mathbf{r}}_i = \sum_{i=1}^{p} \mathbf{f}_i \tag{e}$$

We define the total mass of the system as the sum of the individual masses,

$$m = \sum_{i=1}^{p} m_i^{(p)} \tag{8.3}$$

and the *center of mass,* or *centroid,* of the system of particles as

$$\mathbf{r} = \frac{1}{m} \sum_{i=1}^{p} m_i^{(p)}\mathbf{r}_i \tag{8.4}$$

where $\vec{r}$ is the vector from the origin to the center of mass. Since the total mass and each of the component masses are constant, Eq. 8.4 can be differentiated twice with respect to time to obtain

$$m\ddot{\mathbf{r}} = \sum_{i=1}^{p} m_i^{(p)}\ddot{\mathbf{r}}_i \tag{f}$$

Finally, if the total external force acting on the system of particles is defined as

$$\mathbf{f} = \sum_{i=1}^{p} \mathbf{f}_i \tag{8.5}$$

then Eqs. e and f, and Eq. 8.5 yield

$$m\ddot{\mathbf{r}} = \mathbf{f} \tag{8.6}$$

This result states that the resultant of the external forces on any system of mass equals the total mass of the system times the acceleration of the center of mass. That is, the center of mass moves as if it were a particle of mass m under the action of the force $\mathbf{f}$.

8.3 DYNAMICS OF A BODY

A body can be regarded as a collection of a very large number of particles. In addition, from the definition of rigidity, the location of the particles in a body relative to one another remains unchanged.

In the discussion of the dynamics of a system of particles, the translational equation of motion was derived as

$$\mathbf{f} = m\ddot{\mathbf{r}}$$

Since a body is a particular case of a system of particles, Eq. 8.5 also applies to bodies.

The definition of the center of mass or centroid of a body is found from Eq. 8.4. The summation over the particles is replaced by an integral over the body volume, and the mass of the individual particle is replaced by the infinitesimal mass dm:

$$\mathbf{r} = \frac{1}{m}\int_{(v)} \mathbf{r}^P \, dm \tag{8.7}$$

where $\mathbf{r}^P$ locates an infinitesimal mass, as shown in Fig. 8.3. Vector $\mathbf{r}^P$ is the sum of two vectors:

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s} \tag{8.8}$$

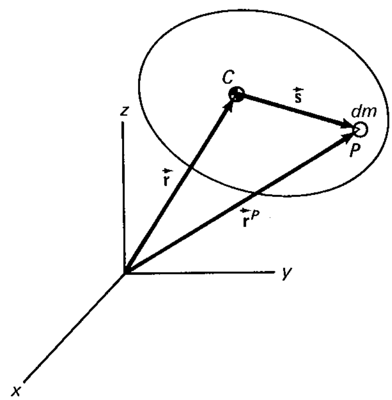

Figure 8.3 A body as a collection of infinitesimal masses.

Substituting Eq. 8.8 into Eq. 8.7 yields

$$\mathbf{r} = \frac{1}{m}\int_{(v)} (\mathbf{r} + \mathbf{s})\, dm$$
$$= \mathbf{r} + \frac{1}{m}\int_{(v)} \mathbf{s}\, dm$$

or

$$\int_{(v)} \mathbf{s}\, dm = \mathit{0} \tag{8.9}$$

The first and second time derivatives of Eq. 8.9 are

$$\int_{(v)} \dot{\mathbf{s}}\, dm = \mathit{0} \tag{8.10}$$

and

$$\int_{(v)} \ddot{\mathbf{s}}\, dm = \mathit{0} \tag{8.11}$$

Equations 8.9 through 8.11 will become useful in the following sections in deriving the equations of motion.

In addition to its tendency to move a body in the direction of its application, a force also tends to rotate the body about any axis that does not intersect the line of action of the force and which is not parallel to it. The measure of this tendency is known as the *moment* of the force about the given axis. The moment of a force is also frequently referred to as *torque*. The rotational equation of motion for a body will be derived in Sec. 8.3.2.

8.3.1 Moments and Couples

Consider a force $\vec{f}_i$ acting on a body, as shown in Fig. 8.4, and a point O not on the line of action of the force. If a vector $\vec{s}$ is introduced from O to any point on the line of action of $\vec{f}_i$, the moment $\vec{n}_i^O$ is found from the vector product:

$$\mathbf{n}_i^O = \tilde{\mathbf{s}}\mathbf{f}_i \tag{8.12}$$

The *principle of moments* is easily proved by applying the distributive law for the sum of vector products. A system of forces $\vec{f}_i$, $i = 1, \ldots, k$, is shown in Fig. 8.5(a) concur-

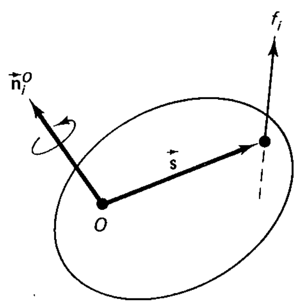

Figure 8.4 The moment of a force $\vec{f}_i$ about point O.

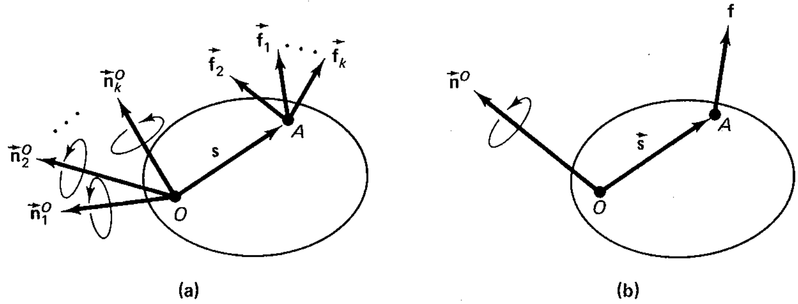

Figure 8.5 (a) A force system acting at point A. (b) The equivalent system.

rent at point A whose position vector from point O is $\vec{s}$. The sum of moments about O is found to be

$$\mathbf{n}^O = \tilde{\mathbf{s}}\mathbf{f} \tag{8.13}$$

where $n^O = \sum_{i=1}^{k} \mathbf{n}_i^O$ and $\mathbf{f} = \sum_{i=1}^{k} \mathbf{f}_i$. Thus the sum of the moments of a system of concurrent forces (forces all of which act at a point) about a given point is equal to the moment of their sum about the same point, as shown in Fig. 8.5 (b).

The moment produced by two equal and opposite and noncollinear forces is known as a *couple*. Couples have certain unique properties. Figure 8.6 shows two equal and opposite forces $\vec{f}$ and $-\vec{f}$ acting on a body. The vector $\vec{s}$ joins *any* point B on the line of action of $-\vec{f}$ to *any* point A on the line of action of $\vec{f}$. Points A and B are located by position vectors $\vec{s}^A$ and $\vec{s}^B$, from *any* point O. The combined moment of the two forces about O is

$$\begin{aligned}\mathbf{n} &= \tilde{\mathbf{s}}^A\mathbf{f} + \tilde{\mathbf{s}}^B(-f)\\ &= (\tilde{\mathbf{s}}^A - \tilde{\mathbf{s}}^B)\mathbf{f}\end{aligned}$$

Since $\mathbf{s}^A - \mathbf{s}^B = \mathbf{s}$, the moment of the couple becomes

$$\mathbf{n} = \tilde{\mathbf{s}}\mathbf{f} \tag{8.14}$$

Thus, the moment of a couple is *the same about all points*. Note that the magnitude of $\mathbf{n}$ is $n = fd$, where d is the perpendicular distance between the lines of action of the two

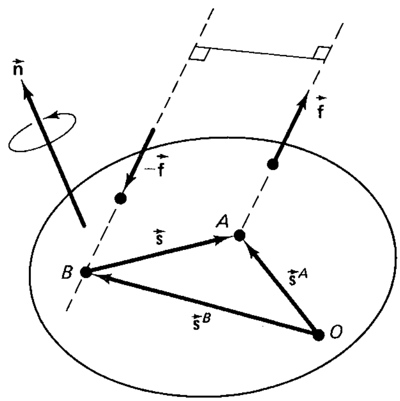

Figure 8.6 The moment of a couple.

forces. It is clear that the moment of a couple is a *free vector*, whereas the moment of a force about a point, which is also the moment about a defined axis through the point, is a *sliding vector* whose direction is along the axis through the point.

The effect of a force acting on a body has been described in terms of its tendency to move the body in the direction of the force and to rotate the body about any axis that does not intersect the line of the force. The representation of this dual effect is often facilitated by replacing the given force by an equal parallel force and a couple to compensate for the change in the moment of the force. This resolution of a force into a force and a couple is illustrated in Fig. 8.7, where the couple has a magnitude $n = fd$.

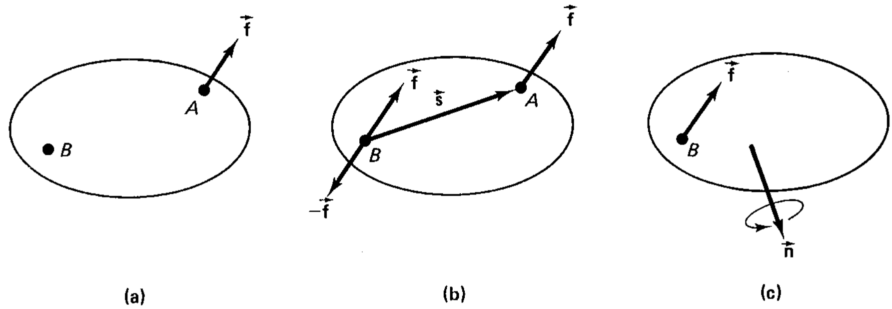

Figure 8.7 (a) A force $\vec{f}$ acting at point A. (b) Two forces $\vec{f}$ and $-\vec{f}$ acting at point B have no effect on the body, since they cancel each other. (c) The couple $\vec{f}$ acting at A and $-\vec{f}$ acting at B are replaced by the moment $\vec{n}$.

To study the motion of a body, it is often convenient to replace the forces acting on the body by an equivalent system consisting of one force and one couple, as shown in Fig. 8.8. The force $\mathbf{f}$ that acts at the centroid of the body is equal to the sum of all k forces acting on the body; i.e.,

$$\mathbf{f} = \sum_{i=1}^{k} \mathbf{f}_i \tag{8.15}$$

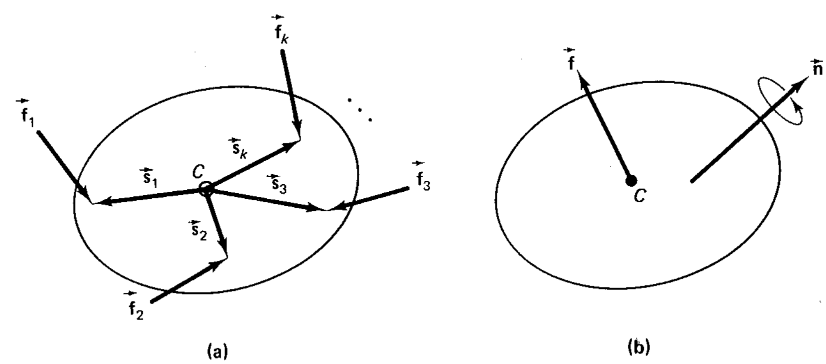

Figure 8.8 (a) A system of forces acting on a body. (b) Equivalent force-couple system.

The moment of the couple is the sum of the moments of the individual forces with respect to the centroid C; i.e.,

$$\mathbf{n} = \sum_{i=1}^{k} \mathbf{n}_i = \sum_{i=1}^{k} \tilde{\mathbf{s}}_i \mathbf{f}_i \tag{8.16}$$

Note that f is acting at the centroid but $\mathbf{n}$ is a free vector.

8.3.2 Rotational Equations of Motion

When the origin of the body-fixed $\xi\eta\zeta$ coordinate system is located at the center of mass of a body, the system is what is known as a *centroidal coordinate system*. Consider the body shown in Fig. 8.9, where the only external force is a force $\vec{f}$ acting on the ith particle of the body. This particle is located with respect to the xyz coordinate system by

$$\mathbf{r}_i^P = \mathbf{r} + \mathbf{s}_i \tag{a}$$

or, by expanding the vectors of both sides of Eq. a into skew-symmetric matrices yields

$$\tilde{\mathbf{r}}_i^P = \tilde{\mathbf{r}} + \tilde{\mathbf{s}}_i \tag{b}$$

Postmultiplying this equation by $\mathbf{f}$ yields

$$\tilde{\mathbf{r}}_i^P \mathbf{f} = \tilde{\mathbf{r}}\mathbf{f} + \tilde{\mathbf{s}}_i \mathbf{f} \tag{c}$$

or

$$\mathbf{n}^G = \tilde{\mathbf{r}}\mathbf{f} + \mathbf{n} \tag{8.17}$$

where $\mathbf{n}^G$ and $\mathbf{n}$ are the moments of $\mathbf{f}$ with respect to the origin of the xyz coordinate system and the body centroid, respectively.

In addition to the external force $\vec{f}$ acting on the ith particle, there are internal forces acting between the particles, such as $\vec{f}_{ij}$ acting on the ith particle by the jth particle. The equation of motion for the ith particle can be written as

$$\mathbf{f} + (\mathbf{f}_{ij} + \cdots) = m_i^{(p)} \ddot{\mathbf{r}}_i^P \tag{d}$$

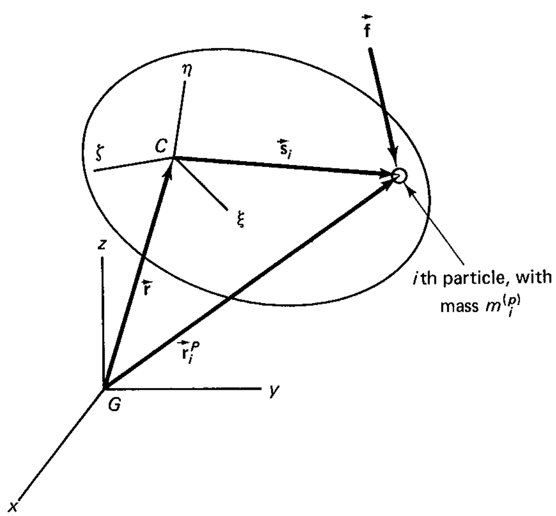

Figure 8.9 A body as a collection of particles under the action of a force $\vec{f}$.

where the terms in the parentheses represent the reaction forces from all other particles of the body acting on the ith particle. Premultiplying Eq. d by $\tilde{\mathbf{r}}_i^P$ results in

$$\tilde{\mathbf{r}}_i^P\mathbf{f} + (\tilde{\mathbf{r}}_i^P\mathbf{f}_{ij} + \cdots) = m_i^{(p)}\tilde{\mathbf{r}}_i^P\ddot{\mathbf{r}}_i^P \tag{e}$$

For the other particles in the body, such as the jth particle, on which we have assumed only reaction forces are acting, the equations of motion have the general form

$$(\mathbf{f}_{ji} + \cdots) = m_j^{(p)}\ddot{\mathbf{r}}_j^P, \qquad j = 1, \ldots, k-1, j \neq i \tag{f}$$

Premultiplying Eq. f by $\tilde{\mathbf{r}}_j^P$ yields

$$(\tilde{\mathbf{r}}_j^P\mathbf{f}_{ji} + \cdots) = m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P, \qquad j = 1, \ldots, k-1, j \neq i \tag{g}$$

Summing Eq. e and all of the $k - 1$ equations in Eq. g results in

$$\tilde{\mathbf{r}}_i^P\mathbf{f} + (\tilde{\mathbf{r}}_i^P\mathbf{f}_{ij} + \tilde{\mathbf{r}}_j^P\mathbf{f}_{ji} + \cdots) = m_i^{(p)}\tilde{\mathbf{r}}_i^P\ddot{\mathbf{r}}_i^P + m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P + \cdots \tag{h}$$

For every $\mathbf{f}_{ij}$, there is an $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$, and therefore Eq. h becomes

$$\mathbf{r}_i^P\mathbf{f} + [(\tilde{\mathbf{r}}_i^P - \tilde{\mathbf{r}}_j^P)\mathbf{f}_{ij} + \cdots] = m_i^{(p)}\tilde{\mathbf{r}}_i^P\ddot{\mathbf{r}}_i^P + m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P + \cdots \tag{i}$$

All vector product terms in the parentheses of Eq. i are identical to zero, since any typical term $(\tilde{\mathbf{r}}_i^P - \tilde{\mathbf{r}}_j^P)\mathbf{f}_{ij}$ is zero (vectors $\mathbf{f}_{ij}$ and $\mathbf{r}_i^P - \mathbf{r}_j^P$ are collinear, as illustrated in Fig. 8.10). Therefore, Eq. i is simplified to

$$\begin{aligned}\tilde{\mathbf{r}}_i^P\mathbf{f} &= m_i^{(p)}\tilde{\mathbf{r}}_i^P\ddot{\mathbf{r}}_i^P + m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P + \cdots \\ &= \sum_{j=1}^{k} m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P\end{aligned}$$

or

$$\mathbf{n}^G = \sum_{j=1}^{k} m_j^{(p)}\tilde{\mathbf{r}}_j^P\ddot{\mathbf{r}}_j^P \tag{j}$$

If the mass of any typical particle is replaced by an infinitesimal mass dm, then the summation can be replaced by an integral over the volume of the body. Hence Eq. j becomes

$$\mathbf{n}^G = \int_{(v)} \tilde{\mathbf{r}}^P\ddot{\mathbf{r}}^P\, dm \tag{k}$$

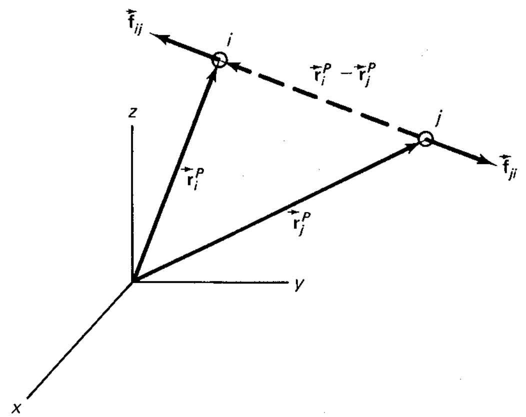

Figure 8.10 Reaction forces between two particles.

Taking $\mathbf{r}^P = \mathbf{r} + \mathbf{s}$ and substituting in Eq. k yields

$$\begin{aligned}\mathbf{n}^G &= \int_{(v)} (\tilde{\mathbf{r}} + \tilde{\mathbf{s}})(\ddot{\mathbf{r}} + \ddot{\mathbf{s}})\, dm \\ &= \tilde{\mathbf{r}}\ddot{\mathbf{r}} \int_{(v)} dm + \int_{(v)} \tilde{\mathbf{s}}\ddot{\mathbf{s}}\, dm \\ &= m\tilde{\mathbf{r}}\ddot{\mathbf{r}} + \int_{(v)} \tilde{\mathbf{s}}\ddot{\mathbf{s}}\, dm \end{aligned} \tag{8.18}$$

where Eqs. 8.9 and 8.11 have been employed.

From Eq. 6.101, i.e., $\dot{\mathbf{s}} = \tilde{\boldsymbol{\omega}}\mathbf{s}$, it is found that

$$\begin{aligned}\ddot{\mathbf{s}} &= \tilde{\dot{\boldsymbol{\omega}}}\mathbf{s} + \tilde{\boldsymbol{\omega}}\dot{\mathbf{s}} \\ &= -\tilde{\mathbf{s}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{s}\end{aligned} \tag{8.19}$$

Premultiplying Eq. 8.19 by $\tilde{\mathbf{s}}$ and rearranging one term yields

$$\tilde{\mathbf{s}}\ddot{\mathbf{s}} = -\tilde{\mathbf{s}}\tilde{\mathbf{s}}\dot{\boldsymbol{\omega}} - \tilde{\mathbf{s}}\tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\boldsymbol{\omega}$$

From Eq. 2.52, it is found that

$$\begin{aligned}\tilde{\mathbf{s}}\tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\boldsymbol{\omega} &= [\tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}} + (\widetilde{\tilde{\mathbf{s}}\boldsymbol{\omega}})]\tilde{\mathbf{s}}\boldsymbol{\omega} \\ &= \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\tilde{\mathbf{s}}\boldsymbol{\omega}\end{aligned}$$

Hence

$$\tilde{\mathbf{s}}\ddot{\mathbf{s}} = -\tilde{\mathbf{s}}\tilde{\mathbf{s}}\dot{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\tilde{\mathbf{s}}\boldsymbol{\omega} \tag{l}$$

Substitution of Eq. l into Eq. 8.18 yields

$$\begin{aligned}\mathbf{n}^G &= m\tilde{\mathbf{r}}\ddot{\mathbf{r}} + \left(-\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\, dm\right)\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\left(-\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\, dm\right)\boldsymbol{\omega} \\ &= \tilde{\mathbf{r}}\mathbf{f} + \mathbf{J}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega}\end{aligned} \tag{8.20}$$

where $\mathbf{J} = -\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\, dm$ is defined as the *global inertia tensor* for the body. Comparison of Eq. 8.17 and 8.20 results in

$$\mathbf{n} = \mathbf{J}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{J}\boldsymbol{\omega} \tag{8.21}$$

Equation 8.21 is the *rotational equation of motion* for a body.

8.3.3 The Inertia Tensor

Figure 8.11 shows a body with its centroidal body-fixed coordinate system. It is assumed that the body has volume v. Vector $\vec{s}$ in the local coordinate system is described as $\mathbf{s}' = [s_{(\xi)}, s_{(\eta)}, s_{(\zeta)}]^T$. The inertia tensor is defined as the integral

$$\mathbf{J}' = -\int_{(v)} \tilde{\mathbf{s}}'\tilde{\mathbf{s}}'\, dm \tag{8.22}$$

which can be written in expanded form as

$$\mathbf{J}' = \int_{(v)} \begin{bmatrix} s_{(\eta)}^2 + s_{(\zeta)}^2 & -s_{(\xi)}s_{(\eta)} & -s_{(\xi)}s_{(\zeta)} \\ -s_{(\eta)}s_{(\xi)} & s_{(\zeta)}^2 + s_{(\xi)}^2 & -s_{(\eta)}s_{(\zeta)} \\ -s_{(\zeta)}s_{(\xi)} & -s_{(\zeta)}s_{(\eta)} & s_{(\xi)}^2 + s_{(\eta)}^2 \end{bmatrix} dm \tag{8.23}$$

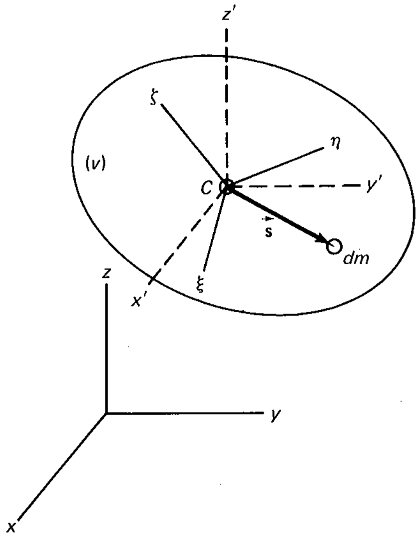

Figure 8.11 A body with centroidal coordinate system.

The following individual integrals are defined:

$$
\begin{aligned}
j_{\xi\xi} &= \int_{(v)} (s_{(\eta)}^2 + s_{(\zeta)}^2)\, dm \\
j_{\eta\eta} &= \int_{(v)} (s_{(\zeta)}^2 + s_{(\xi)}^2)\, dm \\
j_{\zeta\zeta} &= \int_{(v)} (s_{(\xi)}^2 + s_{(\eta)}^2)\, dm \\
j_{\xi\eta} = j_{\eta\xi} &= -\int_{(v)} s_{(\xi)} s_{(\eta)}\, dm \\
j_{\eta\zeta} = j_{\zeta\eta} &= -\int_{(v)} s_{(\eta)} s_{(\zeta)}\, dm \\
j_{\zeta\xi} = j_{\xi\zeta} &= -\int_{(v)} s_{(\zeta)} s_{(\xi)}\, dm
\end{aligned}
\tag{8.24}
$$

where $j_{\xi\xi}$, $j_{\eta\eta}$, and $j_{\zeta\zeta}$ are called the *moments of inertia* and $j_{\xi\eta}$, $j_{\eta\xi}$, $j_{\zeta\eta}$, $j_{\eta\zeta}$, $j_{\zeta\xi}$, and $j_{\xi\zeta}$ are called the *products of inertia*. Then Eq. 8.22 is written as

$$
\mathbf{J}' = \begin{bmatrix} j_{\xi\xi} & j_{\xi\eta} & j_{\xi\zeta} \\ j_{\eta\xi} & j_{\eta\eta} & j_{\eta\zeta} \\ j_{\zeta\xi} & j_{\zeta\eta} & j_{\zeta\zeta} \end{bmatrix}
\tag{8.25}
$$

The matrix $\mathbf{J}'$ is called the *inertia tensor* (inertia matrix) for the body.

If the orientation of the centroidal $\xi\eta\zeta$ body-fixed axes is changed, the moments and products of inertia will change in value. There is one unique orientation of the $\xi\eta\zeta$ axes for which the products of inertia vanish. For this orientation the inertia matrix takes the diagonal form

$$
\mathbf{J}' = \operatorname{diag}[j_{\xi\xi}, j_{\eta\eta}, j_{\zeta\zeta}]
\tag{8.26}
$$

The $\xi\eta\zeta$ axes for which the products of inertia vanish are called the *principal axes of inertia*.

Another form of the global inertia tensor $\mathbf{J}$ as defined in Eq. 8.20 can be derived as follows:

$$\begin{aligned}
\mathbf{J} &= -\int_{(v)} \tilde{\mathbf{s}}\tilde{\mathbf{s}}\, dm \\
&= -\int_{(v)} \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T\mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T\, dm \\
&= \mathbf{A}\left(-\int \tilde{\mathbf{s}}'\tilde{\mathbf{s}}'\, dm\right)\mathbf{A}^T \\
&= \mathbf{A}\mathbf{J}'\mathbf{A}^T
\end{aligned} \tag{8.27}$$

In contrast to $\mathbf{J}'$, which is a constant matrix, $\mathbf{J}$ is a function of the angular orientation of the body.

The time derivatives of $\mathbf{J}'$ and $\mathbf{J}$ are

$$\dot{\mathbf{J}}' = \mathbf{0} \tag{8.28}$$

and

$$\begin{aligned}
\dot{\mathbf{J}} &= \dot{\mathbf{A}}\mathbf{J}'\mathbf{A}^T + \mathbf{A}\mathbf{J}'\dot{\mathbf{A}}^T \\
&= \tilde{\boldsymbol{\omega}}\mathbf{A}\mathbf{J}'\mathbf{A}^T + \mathbf{A}\mathbf{J}'\mathbf{A}^T\tilde{\boldsymbol{\omega}}^T \\
&= \tilde{\boldsymbol{\omega}}\mathbf{J} - \mathbf{J}\tilde{\boldsymbol{\omega}}
\end{aligned} \tag{8.29}$$

where Eq. 6.94 has been employed.

The rotational equations of motion for a body are given by Eq. 8.21 in terms of the global inertia tensor. The equations represented there can be converted to use the local components of the vectors by taking $\mathbf{n} = \mathbf{A}\mathbf{n}'$, $\boldsymbol{\omega} = \mathbf{A}\boldsymbol{\omega}'$, and $\dot{\boldsymbol{\omega}} = \dot{\mathbf{A}}\boldsymbol{\omega}' + \mathbf{A}\dot{\boldsymbol{\omega}}' = \mathbf{A}\tilde{\boldsymbol{\omega}}'\boldsymbol{\omega}' + \mathbf{A}\dot{\boldsymbol{\omega}}' = \mathbf{A}\dot{\boldsymbol{\omega}}'$ and substituting in Eq. 8.21 to get

$$\mathbf{A}\mathbf{n}' = \mathbf{J}\mathbf{A}\dot{\boldsymbol{\omega}}' + \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{A}^T\mathbf{J}\mathbf{A}\boldsymbol{\omega}'$$

Premultiplication by $\mathbf{A}^T$ and application of Eq. 8.27 yield

$$\mathbf{n}' = \mathbf{J}'\dot{\boldsymbol{\omega}}' + \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \tag{8.30}$$

This represents the rotational equations of motion for a body, which are known as Euler's equations of motion.

8.3.4 An Unconstrained Body

Consider the body drawn solid in Fig. 8.12(a), which has no contact with any other body except through force elements. There is no kinematic joint attached to this body to eliminate any of its degrees of freedom. A typical free-body diagram for this body is shown in Fig. 8.12(b). If the sum of all forces acting on the body is denoted by $\mathbf{f}_i$ and the sum of the moment of $\mathbf{f}_i$ and any other pure moments acting on the body is denoted by $\mathbf{n}_i$, then the translational and rotational equations of motion for this body are given, from Eqs. 8.6 and 8.30, as

$$m_i\ddot{\mathbf{r}}_i = \mathbf{f}_i \tag{8.31}$$

$$\mathbf{J}_i'\dot{\boldsymbol{\omega}}_i' + \tilde{\boldsymbol{\omega}}_i'\mathbf{J}_i'\boldsymbol{\omega}_i' = \mathbf{n}_i' \tag{8.32}$$

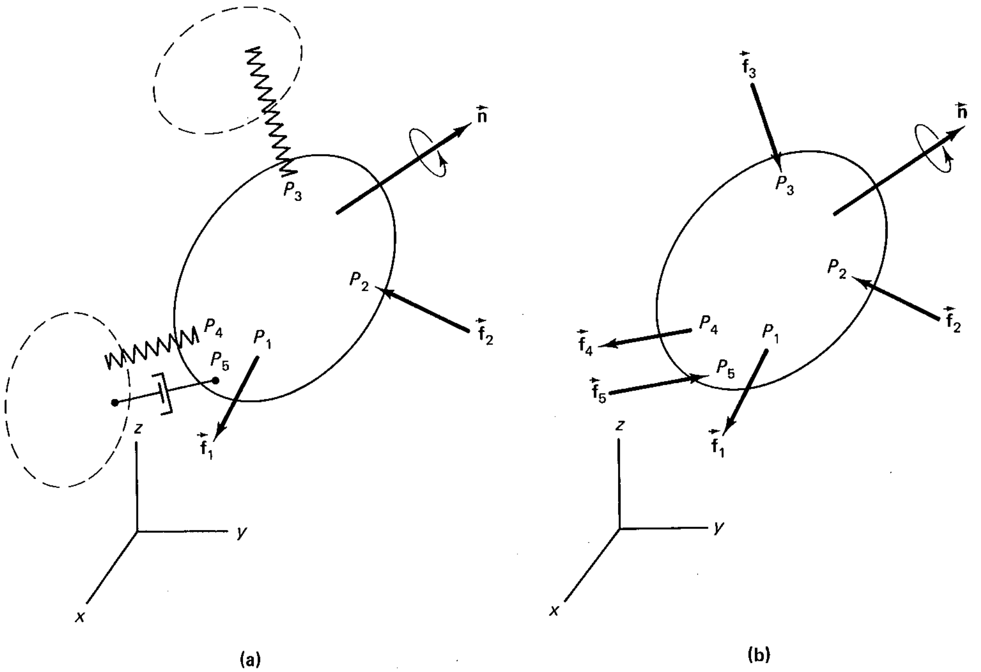

Figure 8.12 (a) An unconstrained body, and (b) its equivalent free-body diagram.

Equations 8.31 and 8.32 are the so-called Newton-Euler equations of motion for an unconstrained body.

Equations 8.31 and 8.32 can be expressed in matrix form as

$$\begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}' \end{bmatrix}_i \begin{bmatrix} \ddot{\mathbf{r}} \\ \dot{\boldsymbol{\omega}}' \end{bmatrix}_i + \begin{bmatrix} \mathit{0} \\ \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{bmatrix}_i = \begin{bmatrix} \mathbf{f} \\ \mathbf{n}' \end{bmatrix}_i \tag{8.33}$$

where

$$\mathbf{N}_i = \operatorname{diag}[m, m, m]_i \tag{8.34}$$

Equation 8.33 can also be written in the compact form

$$\mathbf{M}_i\dot{\mathbf{h}}_i + \mathbf{b}_i = \mathbf{g}_i \tag{8.35}$$

where

$$\mathbf{M}_i = \begin{bmatrix} \mathbf{N} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}' \end{bmatrix}_i \tag{8.36}$$

is the body mass matrix, ~~where~~

$$\mathbf{h}_i = \begin{bmatrix} \dot{\mathbf{r}} \\ \boldsymbol{\omega}' \end{bmatrix}_i \tag{8.37}$$

is the body velocity vector, where

$$\mathbf{b}_i = \begin{bmatrix} \mathit{0} \\ \tilde{\boldsymbol{\omega}}'\mathbf{J}'\boldsymbol{\omega}' \end{bmatrix}_i \tag{8.38}$$

contains the quadratic velocity terms, and where

$$\mathbf{g}_i = \begin{bmatrix} \mathbf{f} \\ \mathbf{n}' \end{bmatrix}_i \tag{8.39}$$

is the body force vector.

8.4 DYNAMICS OF A SYSTEM OF BODIES

A system of bodies making up a mechanical system can be regarded as a collection of individual bodies interconnected by kinematic joints and/or force elements. If there are no kinematic joints in the system, it is called a system of unconstrained bodies. If there are one or more kinematic joints in the system, it is referred to as a system of constrained bodies.

8.4.1 A System of Unconstrained Bodies

Consider the system of unconstrained bodies shown in Fig. 8.13. It is assumed that there are b bodies in this system connected to one another by various force elements. In addition, other forces, either constant or time-dependent, may act on the bodies.

The equations of motion for the ith body were given by Eq. 8.35:

$$\mathbf{M}_i\dot{\mathbf{h}}_i + \mathbf{b}_i = \mathbf{g}_i \tag{a}$$

Equation a can be repeated for $i = 1, \ldots, b$ to obtain

$$\begin{bmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \ddots & \\ & & & \mathbf{M}_b \end{bmatrix} \begin{bmatrix} \dot{\mathbf{h}}_1 \\ \dot{\mathbf{h}}_2 \\ \vdots \\ \dot{\mathbf{h}}_b \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_b \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \vdots \\ \mathbf{g}_b \end{bmatrix} \tag{8.40}$$

or

$$\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g} \tag{8.41}$$

where

$$\mathbf{M} = \text{diag}[\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_b] \tag{8.42}$$

is the system mass matrix, where

$$\mathbf{h} = [\mathbf{h}_1^T, \mathbf{h}_2^T, \ldots, \mathbf{h}_b^T] \tag{8.43}$$

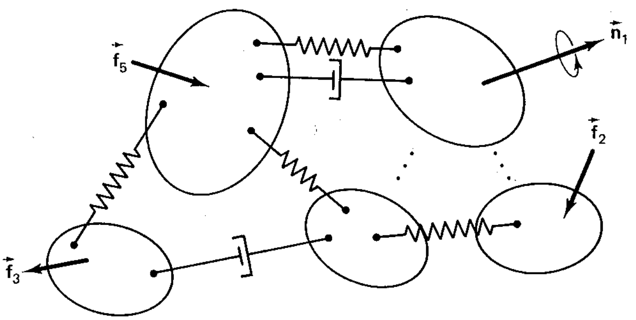

Figure 8.13 A system of unconstrained bodies.

is the system velocity vector, where

$$\mathbf{b} = [\mathbf{b}_1^T, \mathbf{b}_2^T, \ldots, \mathbf{b}_b^T]^T \tag{8.44}$$

contains the quadratic velocity terms, and where

$$\mathbf{g} = [\mathbf{g}_1^T, \mathbf{g}_2^T, \ldots, \mathbf{g}_b^T]^T \tag{8.45}$$

is the system force vector. Equation 8.41 represents the equations of motion for a system of unconstrained bodies. Vector **g** contains all of the *external* and *internal* forces and moments. Gravitational force is considered an external force, whereas the force elements within the system, such as springs, are considered internal forces.†

8.4.2 A System of Constrained Bodies

In a system of constrained bodies, two or more of the bodies are interconnected by kinematic joints. In addition to the kinematic joints, force elements are usually present, as shown in Fig. 8.14.

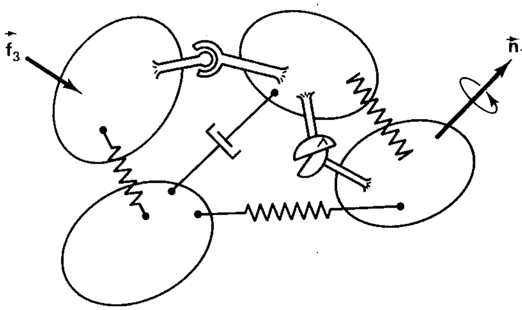

Figure 8.14 A system of constrained bodies.

If the system vector of coordinates for b bodies is denoted by $\mathbf{q}$, then the kinematic joints in the system can be represented as m independent constraints, normally nonlinear equations in terms of $\mathbf{q}$, as:

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{q}) = \mathit{0} \tag{8.46}$$

Each kinematic joint introduces reaction forces between connecting bodies. These reaction forces, which are also referred to as constraint forces, are denoted by vector $\mathbf{g}^{(c)}$:

$$\mathbf{g}^{(c)} = [\mathbf{g}_1^{(c)T}, \mathbf{g}_2^{(c)T}, \ldots, \mathbf{g}_b^{(c)T}]^T \tag{8.47}$$

where $\mathbf{g}_i^{(c)}$, $i = 1, \ldots, b$, is the vector of joint reaction forces acting on body i. The sum of the constraint forces $\mathbf{g}^{(c)}$ and external forces $\mathbf{g}$ provides the total of forces acting on the system. Hence, Eq. 8.41 can be modified to read

$$\mathbf{M}\dot{\mathbf{h}} + \mathbf{b} = \mathbf{g} + \mathbf{g}^{(c)} \tag{8.48}$$

Equations 8.46 and 8.48 together present the equations of motion for a system of constrained bodies.

†In the formulation of the equations of motion, springs, dampers, and actuators are not treated as *bodies* but rather as abstract force elements. This is not an unrealistic assumption, since the mass of these elements is usually negligible as compared to the mass of the connecting bodies.

The constraint force vector $\mathbf{g}^{(c)}$ can be expressed in terms of the constraint equations. However, this task cannot be accomplished at this point, since no rotational coordinates have been defined for the bodies. Equation 8.46 is expressed in terms of the vector of coordinates $\mathbf{q}$, whereas Eq. 8.48 is expressed in terms of the vector of velocities $\mathbf{h}$. For body i the relationship between $\dot{\mathbf{q}}_i$ and $\mathbf{h}_i$ has not yet been defined. The ambiguity lies with the rotational coordinates, not the translational coordinates. Vector $\mathbf{h}_i$ is defined as $\mathbf{h}_i^T = [\dot{\mathbf{r}}^T, \boldsymbol{\omega}'^T]_i^T$, while vector $\mathbf{q}_i$ is $\mathbf{q}^T = [\mathbf{r}^T, ?]_i^T$. This ambiguity will be clarified when proper sets of rotational coordinates for planar and spatial motions are defined in Chaps. 9 and 11.

8.4.3 Constraint Reaction Forces

It is possible to obtain a relationship between the constraint reaction forces and the constraint equations if (1) a proper vector of coordinates is defined and (2) the constraint forces are expressed with respect to the same coordinate system as the vector of coordinates. For example, assume that Euler parameters are used as rotational coordinates; in that case, vector $\mathbf{q}_i = [\mathbf{r}^T, \mathbf{p}^T]_i^T$ is the vector of coordinates for body i. Vector $\dot{\mathbf{q}}_i = [\dot{\mathbf{r}}^T, \dot{\mathbf{p}}^T]_i^T$ is different from vector $\mathbf{h}_i = [\dot{\mathbf{r}}^T, \boldsymbol{\omega}']_i^T$; however, the identity $\boldsymbol{\omega}_i' = 2\mathbf{L}_i\dot{\mathbf{p}}_i$ can transform $\mathbf{h}_i$ to $\dot{\mathbf{q}}_i$ or vice versa. Similarly, for the entire system, $\mathbf{h}$ can be transformed to $\dot{\mathbf{q}}$. The constraint reaction force vector $\mathbf{g}^{(c)}$ and the velocity vector $\mathbf{h}$ are expressed in the same coordinate system. It will be seen in Chap. 11 that vector $\mathbf{g}^{(c)}$ can be transformed to another coordinate system consistent with $\mathbf{q}$.

At this point it will be assumed that $\mathbf{g}^{(c)}$ can be transformed to a coordinate system consistent with $\mathbf{q}$ and denoted as $\mathbf{g}^{(*)}$. It will further be assumed that there are m independent constraint equations

$$\boldsymbol{\Phi} \equiv \boldsymbol{\Phi}(\mathbf{q}) = \boldsymbol{0} \qquad (a)$$

If the joints are assumed to be frictionless, the work[†] done by the constraint forces in a virtual (infinitesimal) displacement[‡] $\delta\mathbf{q}$ is zero; i.e.,

$$\mathbf{g}^{(*)T}\delta\mathbf{q} = 0 \qquad (b)$$

Since the virtual displacement $\delta\mathbf{q}$ must be consistent with the constraints, Eq. a[§] yields

$$\boldsymbol{\Phi}_{\mathbf{q}}\delta\mathbf{q} = \boldsymbol{0} \qquad (c)$$

The vector of n coordinates $\mathbf{q}$ may be partitioned into a set of m dependent coordinates $\mathbf{u}$, and a set of n-m independent coordinates $\mathbf{v}$, as $\mathbf{q} \equiv [\mathbf{u}^T, \mathbf{v}^T]^T$. This yields a partitioned vector of virtual displacements $\delta\mathbf{q} \equiv [\delta\mathbf{u}^T, \delta\mathbf{v}^T]^T$ and a partitioned Jacobian

[†] The work done by a force $\mathbf{f}$ acting on a system and causing a displacement $\mathbf{q}$ is defined as $w = \mathbf{f}^T\mathbf{q}$.

[‡] A virtual displacement of a system is defined as an infinitesimal change in the coordinates of the system consistent with the constraints and forces imposed on the system at time t. The displacement is called a virtual one to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the constraints and forces may change.

[§] The Taylor series expansion of Eq. a about $\mathbf{q}$ is

$$\boldsymbol{\Phi}(\mathbf{q} + \delta\mathbf{q}) = \boldsymbol{\Phi}(\mathbf{q}) + \boldsymbol{\Phi}_{\mathbf{q}}\delta\mathbf{q} + \text{higher-order terms}$$

A displacement $\delta\mathbf{q}$ consistent with the constraints yields $\boldsymbol{\Phi}(\mathbf{q} + \delta\mathbf{q}) = \boldsymbol{0}$. Using $\boldsymbol{\Phi}(\mathbf{q}) = \boldsymbol{0}$ and eliminating the higher-order terms for infinitesimal $\delta\mathbf{q}$, we find that $\boldsymbol{\Phi}_{\mathbf{q}}\delta\mathbf{q} = \boldsymbol{0}$.

matrix $\mathbf{\Phi_q} = [\mathbf{\Phi_u}, \mathbf{\Phi_v}]$. The matrix $\mathbf{\Phi_u}$ is $m \times m$ and nonsingular, since the constraint equations are assumed to be independent. If vector $\mathbf{g}^{(*)}$ is also partitioned as $\mathbf{g}^{(*)} \equiv [\mathbf{g}_{(u)}^{(*)^T}, \mathbf{g}_{(v)}^{(*)^T}]^T$, then Eq. b can be writtten as

$$\mathbf{g}_{(u)}^{(*)^T} \delta\mathbf{u} + \mathbf{g}_{(v)}^{(*)^T} \delta\mathbf{v} = 0$$

or

$$\mathbf{g}_{(u)}^{(*)^T} \delta\mathbf{u} = -\mathbf{g}_{(v)}^{(*)^T} \delta\mathbf{v} \tag{d}$$

Similarly, Eq. c yields

$$\mathbf{\Phi_u}\,\delta\mathbf{u} = -\mathbf{\Phi_v}\,\delta\mathbf{v} \tag{8.49}$$

If Eq. d is appended to the system of equations represented by Eq. 8.49, the result can be written as

$$\begin{bmatrix} \mathbf{g}_{(u)}^{(*)^T} \\ \mathbf{\Phi_u} \end{bmatrix} \delta\mathbf{u} = -\begin{bmatrix} \mathbf{g}_{(v)}^{(*)^T} \\ \mathbf{\Phi_v} \end{bmatrix} \delta\mathbf{v} \tag{e}$$

The matrix to the left in Eq. e is an $(m + 1) \times m$ matrix. Since $\mathbf{\Phi_u}$ is an $m \times m$ nonsingular matrix, the first row of the $(m + 1) \times m$ matrix, i.e., $\mathbf{g}_{(u)}^{(*)^T}$, can be expressed as a linear combination of the other rows of the matrix:

$$\mathbf{g}_{(u)}^{(*)} = \mathbf{\Phi_u}^T \boldsymbol{\lambda} \tag{f}$$

where $\boldsymbol{\lambda}$ is an m-vector of multipliers known as *Lagrange multipliers*. Substitution of Eq. f in Eq. d yields

$$\boldsymbol{\lambda}^T \mathbf{\Phi_u}\,\delta\mathbf{u} = -\mathbf{g}_{(v)}^{(*)^T} \delta\mathbf{v}$$

or

$$-\boldsymbol{\lambda}^T \mathbf{\Phi_v}\,\delta\mathbf{v} = -\mathbf{g}_{(v)}^{(*)^T} \delta\mathbf{v} \tag{g}$$

where Eq. 8.49 has been employed. Vector $\delta\mathbf{v}$ is an arbitrary (independent) vector. The consistency of the constraints for virtual displacements $\delta\mathbf{q}$ is guaranteed by solving Eq. 8.49 for $\delta\mathbf{u}$. Since Eq. g must hold for any arbitrary $\delta\mathbf{v}$, then

$$\boldsymbol{\lambda}^T \mathbf{\Phi_v} = \mathbf{g}_{(v)}^{(*)^T}$$

or

$$\mathbf{g}_{(v)}^{(*)} = \mathbf{\Phi_v}^T \boldsymbol{\lambda} \tag{h}$$

Appending Eq. f to Eq. h yields

$$\mathbf{g}^{(*)} = \mathbf{\Phi_q}^T \boldsymbol{\lambda} \tag{8.50}$$

Equation 8.50 expresses the constraint reaction forces in terms of the constraint equations and a vector of multipliers.

8.5 CONDITIONS FOR PLANAR MOTION

The equations of motion for an unconstrained body can be simplified if the motion is planar. Assume, without any loss of generality, that the centroidal body-fixed $\xi_i\eta_i\zeta_i$ coordinates are attached to body i in such a way that the ζ axis is parallel to the z axis. Furthermore, assume that the center of mass of the body is located in the xy plane. The vector of translational coordinates for the body is then $\mathbf{r}_i = [x, y, 0]_i^T$. It can be assumed that the $\xi\eta$ and xy planes remain coincident, and therefore that $\ddot{z}_i = 0$ at all times. If the angle ϕ_i is taken as the rotational coordinate, then vectors of angular velocity and angular acceleration for the body can be written, respectively, as $\boldsymbol{\omega}_i = \boldsymbol{\omega}'_i = [0, 0, \dot{\phi}]_i^T$ and $\dot{\boldsymbol{\omega}}_i = \dot{\boldsymbol{\omega}}'_i = [0, 0, \ddot{\phi}]_i^T$.

Assume that a force $\vec{f}_i$ is acting at point P_i, as shown in Fig. 8.15. The translational equation of motion given in Eq. 8.31 can be written, where $\ddot{z}_i = 0$, as

$$\begin{bmatrix} m & & \\ & m & \\ & & m \end{bmatrix}_i \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix}_i = \begin{bmatrix} f_{(x)} \\ f_{(y)} \\ f_{(z)} \end{bmatrix}_i \tag{8.51}$$

which yields $f_{(z)_i} = 0$.

Condition 1. In planar motion, the forces acting on a body must remain parallel to the plane of the body.

If it is assumed that P_i is in the plane, then the moment $\mathbf{n}_i = \tilde{\mathbf{s}}_i^P \mathbf{f}_i$ can have a nonzero component only in the z (or ζ) direction: $\mathbf{n}_i = \mathbf{n}'_i = [0, 0, n]_i^T$, where n_i is the magnitude of the moment. Hence, the system of rotational equations of motion, given in Eq. 8.32, is written as

$$\mathbf{J}'_i \begin{bmatrix} 0 \\ 0 \\ \ddot{\phi} \end{bmatrix}_i + \begin{bmatrix} 0 & -\dot{\phi} & 0 \\ \dot{\phi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_i \mathbf{J}'_i \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}_i$$

or, by using the elements of $\mathbf{J}'$ from Eq. 8.25, it is found that

$$\begin{aligned} j_{\xi\zeta_i}\ddot{\phi}_i - j_{\eta\zeta_i}\dot{\phi}_i^2 &= 0 && (a) \\ j_{\eta\zeta_i}\ddot{\phi}_i + j_{\xi\zeta_i}\dot{\phi}_i^2 &= 0 && (b) \\ j_{\zeta\zeta_i}\ddot{\phi}_i &= n_i && (8.52) \end{aligned}$$

Equations a and b can be written as

$$\begin{bmatrix} j_{\xi\zeta} & -j_{\eta\zeta} \\ j_{\eta\zeta} & j_{\xi\zeta} \end{bmatrix}_i \begin{bmatrix} \ddot{\phi} \\ \dot{\phi}^2 \end{bmatrix}_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{c}$$

In order to have a nontrivial solution for Eq. c, i.e., nonzero $\ddot{\phi}_i$ and $\dot{\phi}_i$, the matrix at the left in Eq. c must be singular. Therefore, $j_{\xi\zeta_i}^2 + j_{\eta\zeta_i}^2 = 0$, which yields

$$j_{\xi\zeta_i} = j_{\eta\zeta_i} = 0 \tag{8.53}$$

Condition 2. A moment about the ζ axis causes the body to rotate only about that axis if the products of inertia $j_{\xi\zeta}$ and $j_{\eta\zeta}$ are zero.

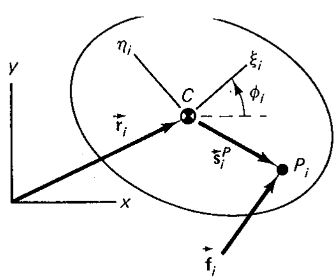

Figure 8.15 A body with centroidal coordinate system in planar motion.

The condition shown by Eq. 8.53 applies to bodies in which the plane of motion is geometrically the plane of symmetry, where uniform distribution of mass is assumed. However, if either of the products of inertia $j_{\xi\zeta}$ and $j_{\eta\zeta}$ is nonzero, a moment about the ζ axis causes the body to rotate about an axis nonparallel to ζ.

An unconstrained body can experience a nonplanar motion if conditions 1 and 2 are not met. However, for a constrained body, kinematic joints may be aligned in such a way that the body would move only in a plane, without conditions 1 and 2 being satisfied. In such cases, if condition 1 is not met, joint reaction forces are developed that are not in the plane of motion, and if condition 2 is not met, joint reaction moments are developed that are not about the axis perpendicular to the plane of motion.