7

Spatial Kinematics

This chapter derives the spatial kinematic constraint equations for several standard kinematic pairs. Those for other standard pairs, or for special-purpose kinematic pairs, may be formulated similarly. Euler parameters are used to define the angular orientation of bodies. The methodology will remain the same if other sets of rotational coordinates are used. However, the quadratic nature of the transformation matrix, the absence of trigonometric functions, and the singularity-free aspect of the Euler parameters make them more attractive than other sets of rotational coordinates. Another advantage of Euler parameter formulation is that it allows kinematic relationships for different pairs to be written in compact matrix form, so that compact and efficient computational algorithms can be developed.

7.1 RELATIVE CONSTRAINTS BETWEEN TWO VECTORS

In this section algebraic relations between two vectors are derived to provide the basis for subsequent constraint equation formulation. Most kinematic constraints require that two vectors remain parallel or perpendicular. A vector may have fixed length, e.g., if it connects points that are fixed in the same body; or it may have variable length, if, for example, it connects points that are fixed in different bodies.

In constraint equation formulation, it is necessary to express the components of all vectors in the same coordinate system, the most natural being the global coordinate system. The global components of a vector that is fixed in a body may be obtained from the vector's local components or they may be obtained from the global coordinates of its endpoints.

Vector $\vec{s}_i$ in Fig. 7.1 is fixed in body i. Thus, its magnitude is constant and its orientation relative to the $\xi_i\eta_i\zeta_i$ axes does not change. The global components of $\vec{s}_i$ can be obtained from any of the following:

$$\begin{aligned} \mathbf{s}_i &= \mathbf{s}_i^B - \mathbf{s}_i^C \\ &= \mathbf{A}_i\mathbf{s}_i'^B - \mathbf{A}_i\mathbf{s}_i'^C \\ &= \mathbf{A}_i(\mathbf{s}_i'^B - \mathbf{s}_i'^C) \end{aligned} \tag{7.1}$$

where $\mathbf{s}_i'^B = [\xi^B, \eta^B, \zeta^B]_i^T$ and $\mathbf{s}_i'^C = [\xi^C, \eta^C, \zeta^C]_i^T$ are known constant quantities. It can be observed from Eq. 7.1 that the global components of a vector that is fixed in body i do not depend on the global location of the body, i.e., on the vector $\mathbf{r}_i$.

When a vector connects two points on bodies i and j, such as vector $\vec{d}$ in Fig. 7.1, its global components are written as

$$\begin{aligned} \mathbf{d} &= (\mathbf{r}_j + \mathbf{s}_j^B) - (\mathbf{r}_i + \mathbf{s}_i^B) \\ &= \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^B - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^B \end{aligned} \tag{7.2}$$

where $\mathbf{s}_j'^B = [\xi^B, \eta^B, \zeta^B]_j^T$ is constant. It is clear that the global components of a vector that connects points on two bodies depend on the global position of the bodies, i.e., on vectors $\mathbf{r}_i$ and $\mathbf{r}_j$.

In the following subsections, constraint equations are derived by imposing conditions between vectors in adjacent bodies. In general, either the vectors are of constant magnitude and are embedded in different bodies, e.g., $\vec{s}_i$ and $\vec{s}_j$ in Fig. 7.1; or one vector is embedded in one body and the other vector is connected between points on it and an adjacent body and may have fixed or variable magnitude, e.g., vectors $\vec{s}_i$ and $\vec{d}$ or $\vec{s}_j$ and $\vec{d}$. If we write a constraint between two vectors having constant magnitudes, then we will refer to it as *type 1* constraint. If the constraint is between two vectors, one having fixed magnitude and the other being variable, then we will refer to it as *type 2* constraint. Constraint equations in this chapter are assigned a superscript with two indices. The first index denotes the type of constraint, and the second index defines the number of independent equations in the expression.

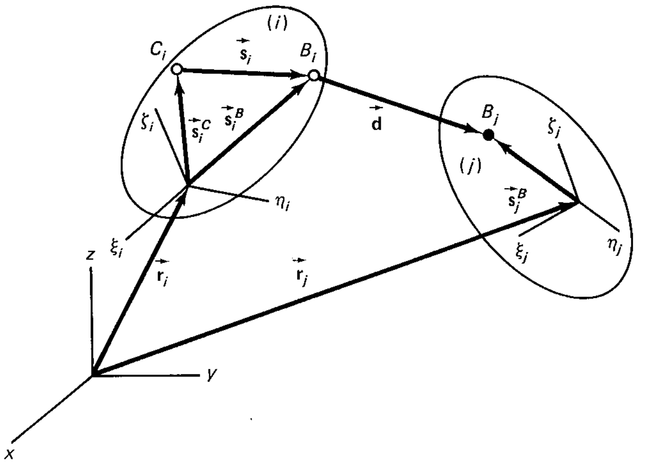


Figure 7.1 Vectors with constant and with varying magnitudes.

7.1.1 Two Perpendicular Vectors

To specify that two vectors must remain perpendicular (normal) at all times, we need one constraint relation. If the direction of one vector is specified, the second vector can translate and rotate only in planes perpendicular to the first vector. Vectors $\vec{s}_i$ and $\vec{s}_j$ shown in Fig. 7.1 remain perpendicular if their scalar product is zero, i.e., if

$$\begin{aligned}\Phi^{(n1,1)} &\equiv \mathbf{s}_i^T\mathbf{s}_j \\ &= \mathbf{s}_i'^T\mathbf{A}_i^T\mathbf{A}_j\mathbf{s}_j' = 0\end{aligned} \tag{7.3}$$

Note that the superscription Φ indicates that this is a normal type 1 constraint having 1 equation.

If vector $\vec{d}$ in Fig. 7.1, which is connected between bodies i and j, is to remain perpendicular to $\vec{s}_i$ (normal type 2), then

$$\begin{aligned}\Phi^{(n2,1)} &\equiv \mathbf{s}_i^T\mathbf{d} \\ &= \mathbf{s}_i'^T\mathbf{A}_i^T(\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^B - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^B) = 0\end{aligned} \tag{7.4}$$

7.1.2 Two Parallel Vectors

For two vectors to remain parallel, two constraint equations are required. The two constraint equations are derived by setting the vector product of the two vectors to zero. The vector product yields three algebraic equations, of which only two are independent; i.e., one of the equations can be derived by combining the remaining two equations. Therefore, two of the equations can serve as the constraint equations.

For two vectors $\vec{s}_i$ and $\vec{s}_j$ that are embedded in corresponding bodies, the constraint equations imposing parallelism (parallel type 1) are

$$\begin{aligned}\mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j \\ &= \mathbf{A}_i\tilde{\mathbf{s}}_i'\mathbf{A}_i^T\mathbf{A}_j\mathbf{s}_j' = \mathbf{0}\end{aligned} \tag{7.5}$$

where Eq. 6.89 has been employed. For a vector $\vec{s}_i$ with constant magnitude and a vector $\vec{d}$, the constraint equations (parallel type 2) are written as

$$\begin{aligned}\mathbf{\Phi}^{(p2,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{d} \\ &= \mathbf{A}_i\tilde{\mathbf{s}}_i'\mathbf{A}_i^T(\mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^B - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^B) = \mathbf{0}\end{aligned} \tag{7.6}$$

Note that $\mathbf{\Phi}^{(p1,2)} = \mathbf{0}$ and $\mathbf{\Phi}^{(p2,2)} = \mathbf{0}$ provide three equations each. However, the sets of equations each have only two independent equations. There exists a critical case that is associated with selection of two equations from Eq. 7.5 or 7.6. The critical case occurs when the two vectors become parallel to one of the global coordinate axes. To show how this critical case arises, consider Eq. 7.5 in component form:

$$-s_{(z)i}s_{(y)j} + s_{(y)i}s_{(z)j} = 0 \tag{a}$$

$$s_{(z)i}s_{(x)j} - s_{(x)i}s_{(z)j} = 0 \tag{b}$$

$$-s_{(y)i}s_{(x)j} + s_{(x)i}s_{(y)j} = 0 \tag{c}$$

where $\mathbf{s}_i \equiv [s_{(x)}, s_{(y)}, s_{(z)}]_i^T$ and $\mathbf{s}_j \equiv [s_{(x)}, s_{(y)}, s_{(z)}]_j^T$. Assume as an example that vectors $\vec{s}_i$ and $\vec{s}_j$ are located on bodies i and j of a cylindrical joint as shown in Fig. 7.2. At time t the joint axis, and hence vectors $\vec{s}_i$ and $\vec{s}_j$, has become parallel to the z coordinate axis.

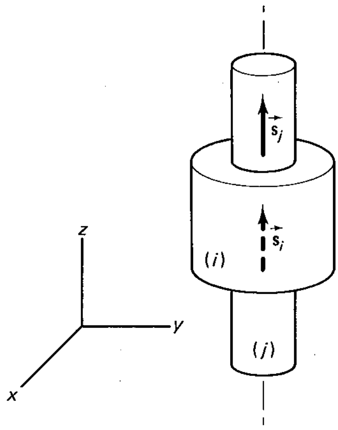


Figure 7.2 The axis of the cylindrical joint may become parallel to one of the coordinate axes, e.g., the z axis.

Then $s_{(x)i} = s_{(y)i} = 0$ and $s_{(x)j} = s_{(y)j} = 0$. It is clear that Eqs. a, b, and c are satisfied. However, for position, velocity, and acceleration analysis the Jacobian matrix of the constraint equations is needed. In this matrix, the row associated with Eq. c appears as

$$
\begin{array}{c} \\ \partial(\text{Eq. } c)/\partial \cdots \\ \\ \end{array}
\begin{bmatrix}
 & \mathbf{p}_i & & \mathbf{p}_j & \\
 & \vdots & & \vdots & \\
\cdots & \textcircled{1} & \cdots\cdots & \textcircled{2} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
$$

The possible nonzero entries ① and ② are

$$
\textcircled{1} = -\frac{\partial s_{(y)i}}{\partial \mathbf{p}_i} s_{(x)j} + \frac{\partial s_{(x)i}}{\partial \mathbf{p}_i} s_{(y)j}
$$

$$
\textcircled{2} = -\frac{\partial s_{(x)j}}{\partial \mathbf{p}_j} s_{(y)i} + \frac{\partial s_{(y)j}}{\partial \mathbf{p}_j} s_{(x)i}
$$

These entries for this example are identical to zero, and hence Eq. c leads to a reduction in the row rank of the Jacobian matrix and to numerical difficulties. Therefore, Eqs. a and b must be selected, since both contain the nonzero components $s_{(z)i}$ and $s_{(z)j}$. A technique for the selection of a proper set of equations can be stated as follows:

> *Compare the absolute values of* $s_{(x)i}$, $s_{(y)i}$, *and* $s_{(z)i}$ *and select the two equations (out of three) having the largest terms.*

7.2 RELATIVE CONSTRAINTS BETWEEN TWO BODIES

In the following subsections constraint equations for several commonly used lower-pair kinematic joints are derived. These equations fall under the category of holonomic constraints.

7.2.1 Spherical, Universal, and Revolute Joints (LP)

A *spherical* or *ball joint* between two adjacent bodies i and j is shown in Fig. 7.3. The center of the spherical joint, point P, has constant coordinates with respect to the $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ coordinate systems. There are three algebraic equations for this joint; they can be found from the vector equation $\vec{r}_i + \vec{s}_i^P - \vec{s}_j^P - \vec{r}_j = \vec{0}$, as follows:

$$\mathbf{\Phi}^{(s,3)} \equiv \mathbf{r}_i + \mathbf{A}_i\mathbf{s}_i'^P - \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_j = \mathbf{0} \tag{7.7}$$

There are three relative degrees of freedom between two bodies that are connected by a spherical joint.

A *universal* or *Hooke joint* between bodies i and j is shown in Fig. 7.4(a). One bar of the cross can be considered part of body i and the other bar can be considered an extension of body j. Point P, the intersection of the axes of the bars, has constant coordinates with respect to both body-fixed coordinate systems. Therefore, at point P, the constraint of Eq. 7.7 can be applied. The remaining constraint is that the two vectors $\vec{s}_i$ and $\vec{s}_j$, arbitrarily placed on the cross axes, remain perpendicular. Therefore, the constraint equations for a universal joint are

$$\begin{aligned} \mathbf{\Phi}^{(s,3)} &= \mathbf{0} \\ \Phi^{(n1,1)} &\equiv \mathbf{s}_i^T\mathbf{s}_j = 0 \end{aligned} \tag{7.8}$$

There are two relative degrees of freedom between a pair of bodies that are connected by a universal joint.

The constraint formulation of Eq. 7.8 is for the general case of a universal joint between two bodies. This formulation can be simplified for special cases. For example, consider the configuration in Fig. 7.4(b), where body-fixed coordinates are embedded in the bodies in such a way that the ξ_i axis and ζ_j axis are parallel to their corresponding bars of the cross and therefore ξ_i and ζ_j must remain perpendicular. Two unit vectors $\mathbf{u}_i' = [1,0,0]^T$ and $\mathbf{u}_j' = [0,0,1]^T$ can be defined on the ξ_i and ζ_j axes, respectively. Hence, $\mathbf{u}_i = \mathbf{A}_i\mathbf{u}_i'$ and $\mathbf{u}_j = \mathbf{A}_j\mathbf{u}_j'$ form one constraint equation,

$$\Phi^{(n1,1)} \equiv \mathbf{u}_i^T\mathbf{u}_j = 0 \tag{7.9}$$

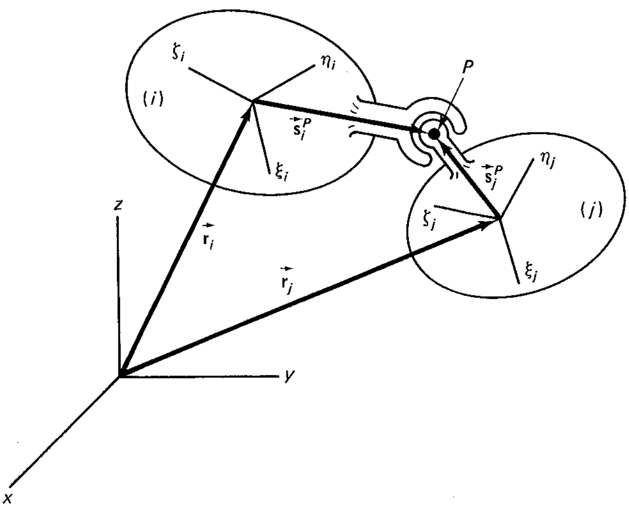


Figure 7.3 A spherical joint.

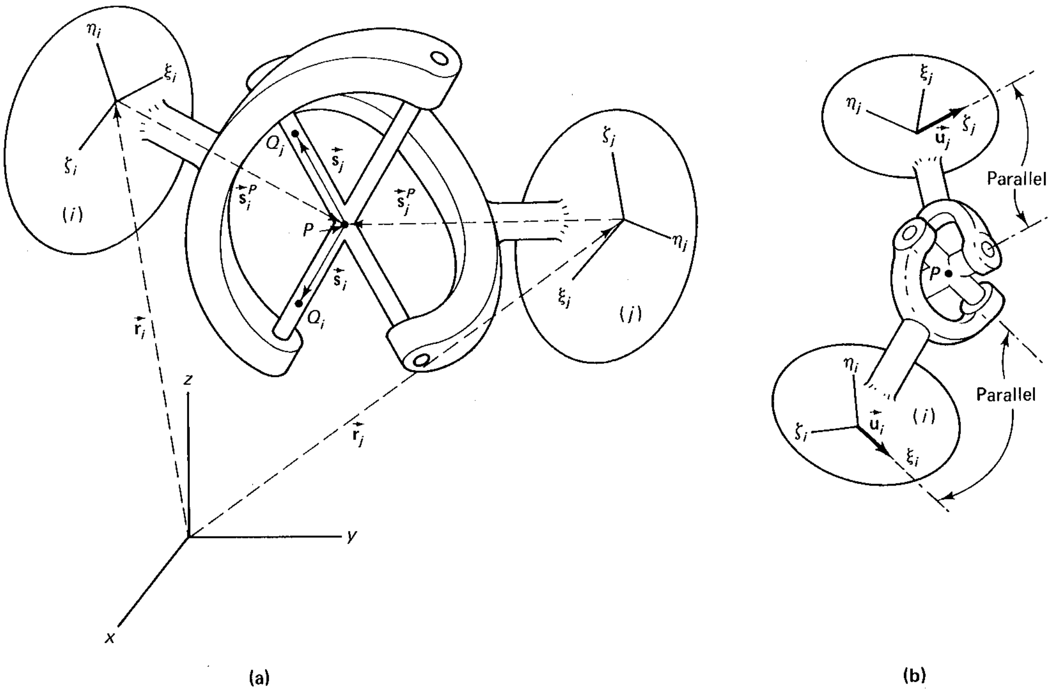


Figure 7.4 A universal joint: (a) general case, and (b) special case where ξ_i axis and ζ_j axis are parallel to their corresponding bars of the cross.

that replaces the constraint $\mathbf{s}_i^T\mathbf{s}_j = 0$ of Eq. 7.8. For this special case, only point P needs to be defined—there is no need to define points Q_i and Q_j.

A *revolute joint* between bodies i and j is shown in Fig. 7.5(a). Any point on the revolute-joint axis has constant coordinates in both local coordinate systems. Equation 7.7 can be imposed on an arbitrary point P on the joint axis. Two other points, Q_i on body i and Q_j on body j, are also chosen arbitrarily on the joint axis. It is clear that vectors $\vec{s}_i$ and $\vec{s}_j$ must remain parallel. Therefore, there are five constraint equations for a revolute joint:

$$\begin{aligned} \boldsymbol{\Phi}^{(s,3)} &= \boldsymbol{0} \\ \boldsymbol{\Phi}^{(p1,2)} &= \tilde{\mathbf{s}}_i\mathbf{s}_j = \boldsymbol{0} \end{aligned} \tag{7.10}$$

There is only one relative degree of freedom between two bodies connected by a revolute joint.

The constraint formulation for a revolute joint may be simplified for special cases. Consider as an example the configuration in Fig. 7.5(b), where the body-fixed coordinates are placed in such a way that the ζ_i and ζ_j axes are parallel to the revolute-joint axis. In such a case, the two unit vectors $\mathbf{u}_i' = [0, 0, 1]^T$ and $\mathbf{u}_j' = [0, 0, 1]^T$ must remain parallel at all times; i.e.,

$$\boldsymbol{\Phi}^{(p1,2)} \equiv \tilde{\mathbf{u}}_i\mathbf{u}_j = \boldsymbol{0} \tag{7.11}$$

This equation replaces the $\tilde{\mathbf{s}}_i\mathbf{s}_j = \boldsymbol{0}$ constraints in Eq. 7.10. For this or other, similar special cases, only one point on the joint axis (point p) needs to be defined.

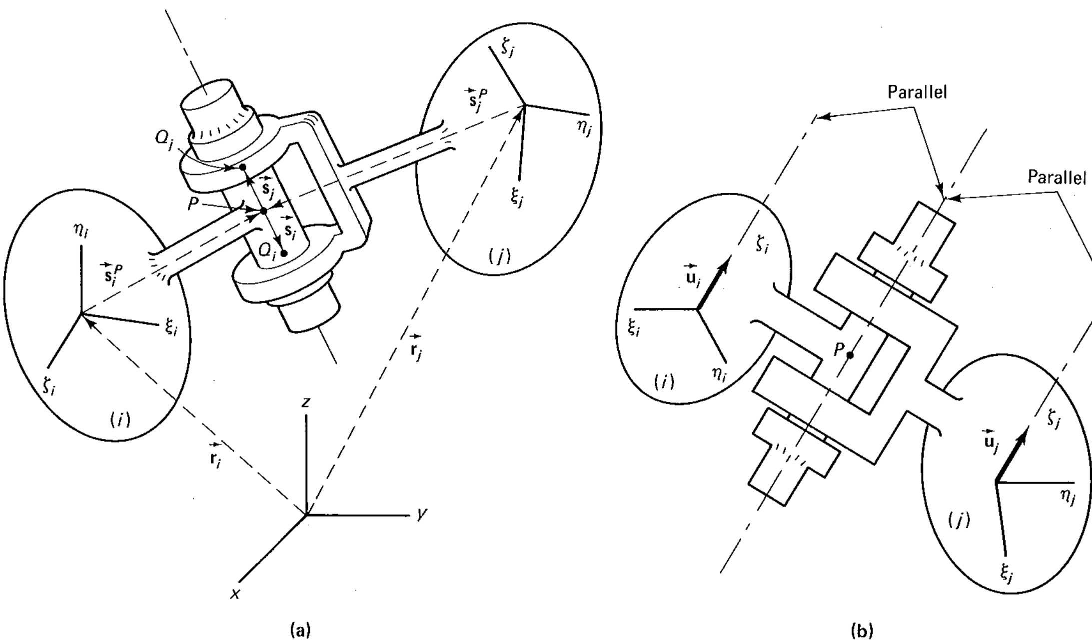


Figure 7.5 A revolute joint: (a) general case, and (b) special case when the ζ_i axis, the ζ_j axis, and the joint axis are parallel.

For the special case of Fig. 7.5(b), another method can be used to keep the ζ_i and ζ_j axes parallel. Since these two axes are parallel to the joint axis, the joint axis is the relative orientational axis of rotation. If a relative set of Euler parameters is defined as $\mathbf{p}_{ij} = [e_0, e_1, e_2, e_3]_{ij}^T$, Eq. 6.122 must hold; i.e.,

$$\mathbf{p}_{ij} = \mathbf{L}_j^* \mathbf{p}_i$$

For this configuration, since the relative axis of rotation is parallel to the ζ axes, $e_{1ij} = e_{2ij} = 0$. Therefore two algebraic equations are obtained:

$$\mathbf{\Phi}^{(ep,2)} \equiv \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \end{bmatrix}_j \mathbf{p}_i = \mathit{0} \qquad (7.12)$$

These two equations may be used instead of the $\mathbf{\Phi}^{(p1,2)}$ constraints of Eq. 7.10.

The special-case configurations of Figs. 7.4(b) and 7.5(b) are not unique. Other special-case configurations may be defined in order to simplify the constraint formulation.

7.2.2 Cylindrical, Translational, and Screw Joints (LP)

A *cylindrical joint* constrains two bodies i and j to move along a common axis, but allows relative rotation about this axis. To derive equations of constraint for this joint, four points, P_i and Q_i on body i and P_j and Q_j on body j, are arbitrarily chosen on the joint axis,† as shown in Fig. 7.6(a). It is required that the vectors $\vec{s}_i$ and $\vec{s}_j$ of constant

†Points defined on a body may not be physically located on that body. For example, points P_i and Q_i are inside the hollow cylinder of body i. They are defined with respect to the $\xi_i\eta_i\zeta_i$ coordinate axes and they move with body i.

magnitude and $\vec{d}$ of variable magnitude remain collinear. Therefore, four constraint equations are needed to define a cylindrical joint; they can be found from two vector product conditions:

$$\begin{aligned} \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j = \mathbf{0} \\ \mathbf{\Phi}^{(p2,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{d} = \mathbf{0} \end{aligned} \tag{7.13}$$

Thus, there are two relative degrees of freedom between bodies connected by a cylindrical joint.

The constraint formulation for cylindrical joints may be simplified in special cases. A special case is shown in Fig. 7.6(b), where the ζ_i and ζ_j axes are parallel to the joint axis. Only two points, P_i and P_j, are placed on the joint axis to define a vector $\vec{d}$. This vector must remain parallel to two unit vectors $\vec{u}_i$ and $\vec{u}_j$. Therefore, in this case only four constraint equations are needed:

$$\begin{aligned} \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{u}}_i\mathbf{u}_j = \mathbf{0} \\ \mathbf{\Phi}^{(p2,2)} &\equiv \tilde{\mathbf{u}}_i\mathbf{d} = \mathbf{0} \end{aligned} \tag{7.14}$$

Another special case with even simpler formulation is shown in Fig. 7.6(c), where the ζ axes coincide with the joint axis. Vector $\mathbf{d} = \mathbf{r}_i - \mathbf{r}_j$ must remain parallel to the unit vectors $\mathbf{u}_i$ and $\mathbf{u}_j$. This can be established by the constraints of Eq. 7.14. For this special case, there is no need to define any arbitrary points on the joint axis.

A *translational* or *prismatic joint* is similar to a cylindrical joint with the exception that the two bodies cannot rotate relative to each other. Therefore, the cylindrical-joint equations apply and one additional equation is required. Two perpendicular vectors, $\vec{h}_i$ and $\vec{h}_j$ on bodies i and j, as shown in Fig. 7.7, must remain perpendicular. Therefore, there are five constraint equations for a translational joint:

$$\begin{aligned} \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j = \mathbf{0} \\ \mathbf{\Phi}^{(p2,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{d} = \mathbf{0} \\ \Phi^{(n1,1)} &\equiv \mathbf{h}_i^T\mathbf{h}_j = 0 \end{aligned} \tag{7.15}$$

The vectors $\vec{h}_i$ and $\vec{h}_j$ are located so they are perpendicular to the line of translation. The relative number of degrees of freedom between two bodies that are connected by a translational joint is 1.

The body-fixed coordinates can be embedded in bodies i and j in special-case configurations, much as in the special cases of cylindrical joints shown in Fig. 7.6(b) and (c). If $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ are parallel and ζ_i and ζ_j are also parallel to the joint axis, then the constraint $\mathbf{h}_i^T\mathbf{h}_j = 0$ of Eq. 7.15 can be replaced by a similar constraint, but without defining any additional points such as R_i and R_j. Since the ξ_i axis is perpendicular to the η_j axis, then unit vectors on these axes must remain perpendicular at all times.

Figure 7.8 illustrates a *screw joint* between bodies i and j, which can rotate and translate about a common axis. However, the rotation and translation are related to each other by the pitch of the screw. To formulate this joint, four constraint equations for an equivalent cylindrical joint can be used. A fifth constraint equation must be supplied, to provide the relation between relative translation and rotation of the bodies. For this purpose, two unit vectors $\vec{u}_i$ and $\vec{u}_j$ perpendicular to the joint axis are embedded in bodies i and j, respectively, as shown in Fig. 7.8. If the initial angle between $\vec{u}_i$ and $\vec{u}_j$ is θ^0 and the instantaneous angle between the two vectors is denoted by θ, then $\theta - \theta^0$ is the relative rotation between the two bodies. Similarly, if the initial magnitude of vector $\vec{d}$ is d^0 and its instantaneous magnitude is denoted by d, then $d - d^0$ is the relative translational

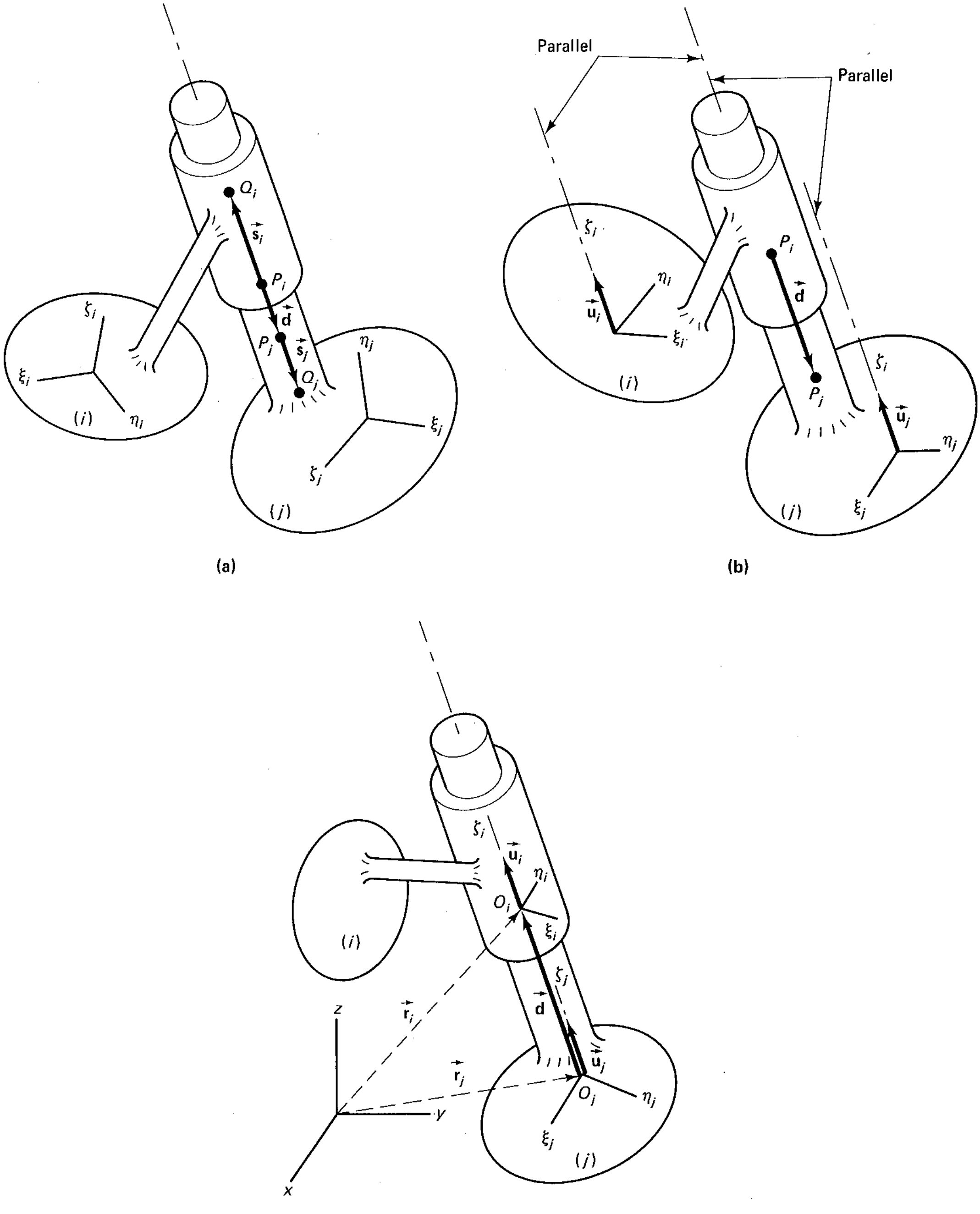


Figure 7.6 A cylindrical joint: (a) general case; (b, c) special cases.

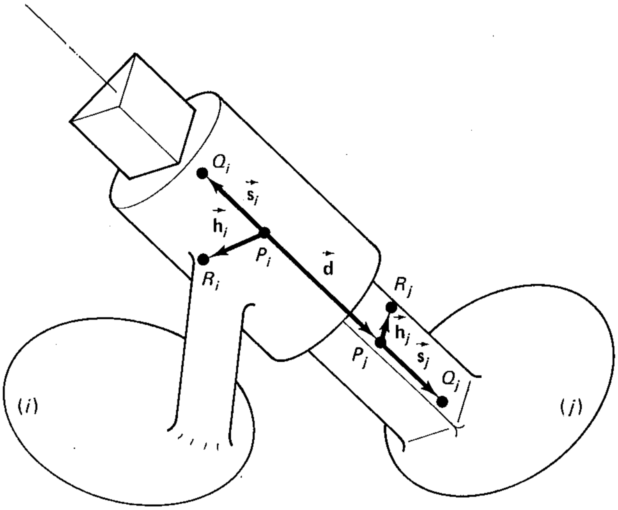


Figure 7.7 A translational joint.

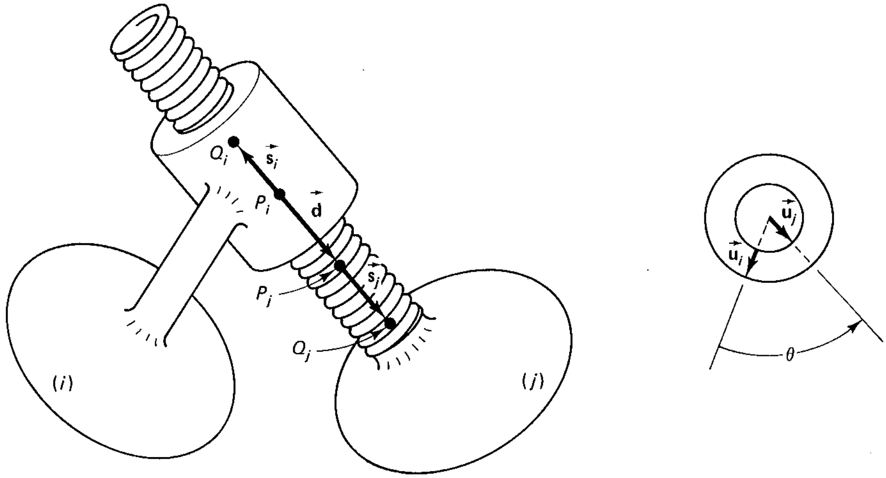


Figure 7.8 A screw joint: (a) side view and (b) top view.

displacement between bodies i and j. Therefore, the five constraint equations can be written as follows:

$$\begin{aligned} \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i \mathbf{s}_j = \mathbf{0} \\ \mathbf{\Phi}^{(p2,2)} &\equiv \tilde{\mathbf{s}}_i \mathbf{d} = \mathbf{0} \\ \Phi^{(r,1)} &\equiv (d - d^0) - \alpha(\theta - \theta^0) = 0 \end{aligned} \tag{7.16}$$

where α is the pitch rate of the screw joint.

The angle θ for a screw joint can be treated as an artificial coordinate. Therefore, one additional constraint equation must be considered with the five constraints of Eq. 7.16:†

$$\Phi^{(\theta,1)} \equiv \mathbf{u}_i^T\mathbf{u}_j - \cos\theta = 0 \tag{7.17}$$

If body-fixed coordinates are embedded in the bodies as was done in the special case of Fig. 7.6(c), then vector $\mathbf{d} = \mathbf{r}_i - \mathbf{r}_j$ is obtained easily. Furthermore, since both ζ axes are parallel to the joint axis, the joint axis is the relative axis of rotation between bodies i and j. Therefore, from Eq. 6.120, we can write one constraint equation,

$$\Phi^{(\theta,1)} \equiv \mathbf{p}_i^T\mathbf{p}_j - \cos\frac{\theta}{2} = 0 \tag{7.18}$$

that can be used instead of Eq. 7.17. In Eq. 7.18, θ is the relative angle of rotation between the bodies, and it is assumed that the two body-fixed coordinates are initially parallel, i.e., that $\theta^0 = 0$.

7.2.3 Composite Joints

Kinematic joints can be combined and modeled as composite joints in order to reduce the number of coordinates and constraint equations. Several examples of such composite joints are shown in this section.

Figure 7.9 illustrates two bodies connected by a rigid link that contains two spherical joints; the entire system is called a *spherical-spherical joint*. Only one constraint equation is required for this joint; it may be written in the form

$$\Phi^{(s\text{-}s,1)} \equiv \mathbf{d}^T\mathbf{d} - l^2 = 0 \tag{7.19}$$

where

$$\mathbf{d} = \mathbf{r}_j + \mathbf{A}_j\mathbf{s}_j'^P - \mathbf{r}_i - \mathbf{A}_i\mathbf{s}_i'^P \tag{7.20}$$

and l is the actual length of the link.

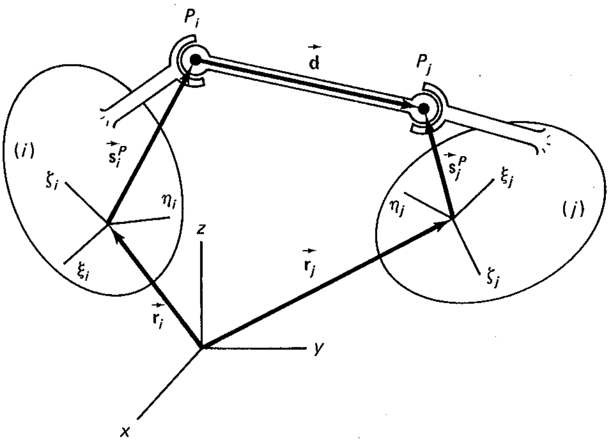


Figure 7.9 A spherical-spherical joint.

†If θ is not treated as an artifical variable, then Eq. 7.17 is not needed. In this case, θ may be calculated from $\theta = \cos^{-1}(\mathbf{u}_i^T\mathbf{u}_j)$ for $0 \le \theta \le \pi$; and when $\pi < \theta < 2\pi$, additional tests are required. This approach is cumbersome and computationally inefficient.

A spherical-revolute joint is shown in Fig. 7.10. Two points P_j and Q_j are defined on the revolute-joint axis in such a way that vector $\vec{d}$ is perpendicular to vector $\vec{s}_j$ as shown. If the distance between points P_i and P_j is to remain equal to l, then two constraint equations are written:

$$\begin{aligned} \Phi^{(s\text{-}s,\,1)} &\equiv \mathbf{d}^T\mathbf{d} - l^2 = 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_j = 0 \end{aligned} \tag{7.21}$$

Two *revolute-revolute* joints are shown in Fig. 7.11. In the configuration of Fig. 7.11(a), the two revolute-joint axes are assumed to be perpendicular. Four constraint equations for this joint may be written, as follows:

$$\begin{aligned} \Phi^{(s\text{-}s,\,1)} &\equiv \mathbf{d}^T\mathbf{d} - l^2 = 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_i = 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_j = 0 \\ \Phi^{(n1,\,1)} &\equiv \mathbf{s}_i^T\mathbf{s}_j = 0 \end{aligned} \tag{7.22}$$

When the revolute-joint axes are parallel, as shown in Fig. 7.11(b), the four constraint equations are

$$\begin{aligned} \Phi^{(s-s,\,1)} &\equiv \mathbf{d}^T\mathbf{d} - l^2 = 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_i = 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_j = 0 \\ \mathbf{\Phi}^{(p1,\,1)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j = \mathbf{0} \end{aligned} \tag{7.23}$$

where $\mathbf{\Phi}^{(p1,1)}$ indicates that only *one out of three* equations of $\mathbf{\Phi}^{(p1)}$ is needed. The reason is that the equations $\mathbf{d}^T\mathbf{s}_i = 0$, $\mathbf{d}^T\mathbf{s}_j = 0$, and only one of the three equations from $\tilde{\mathbf{s}}_i\mathbf{s}_j = \mathbf{0}$ are independent. However, it is possible that one of the constraint equations

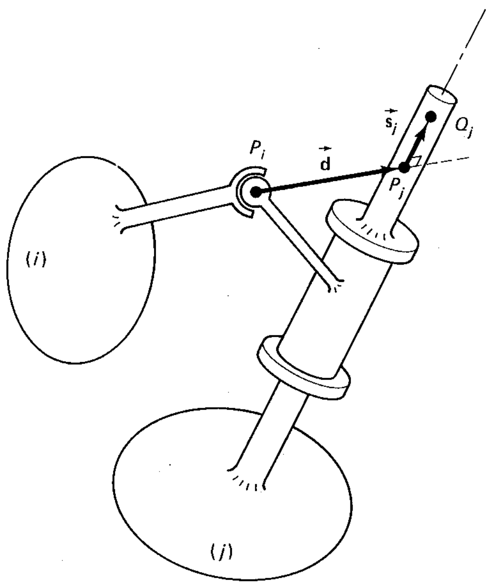


Figure 7.10 A spherical-revolute joint.

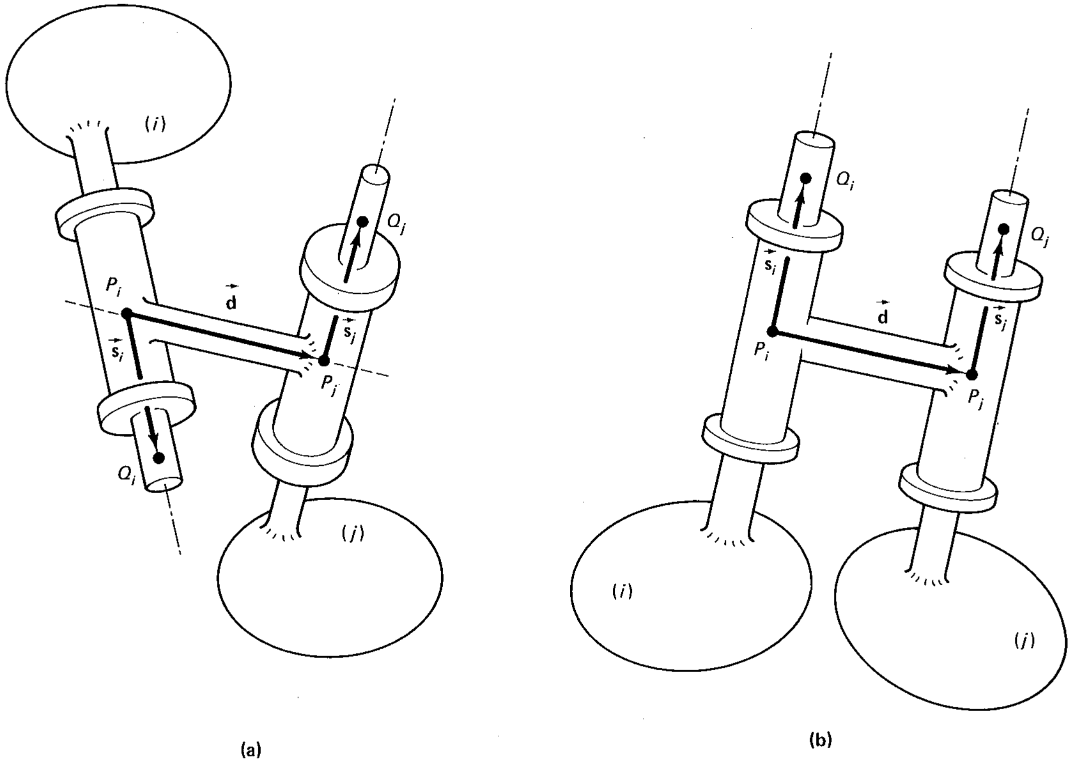


Figure 7.11 Revolute-revolute joint links: (a) perpendicular and (b) parallel.

$\mathbf{d}^T\mathbf{s}_i = 0$ or $\mathbf{d}^T\mathbf{s}_j = 0$ could be ignored, so that two equations from $\mathbf{\Phi}^{(p1,2)}$ could be selected. In this case, either of the following sets may be used:

$$\begin{aligned} \Phi^{(s\text{-}s,\,1)} &= 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_i = 0 \\ \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j = \boldsymbol{0} \end{aligned} \tag{7.24}$$

or

$$\begin{aligned} \Phi^{(s\text{-}s,\,1)} &= 0 \\ \Phi^{(n2,\,1)} &\equiv \mathbf{d}^T\mathbf{s}_j = 0 \\ \mathbf{\Phi}^{(p1,2)} &\equiv \tilde{\mathbf{s}}_i\mathbf{s}_j = \boldsymbol{0} \end{aligned} \tag{7.25}$$

A *revolute-cylindrical joint* that connects two bodies is shown in Fig. 7.12. Formulation of this joint as a revolute joint and a cylindrical joint involving three bodies would require nine constraint equations. This would result in the elimination of a total of 9 degrees of freedom, leaving 3 relative degrees of freedom between bodies i and j. This joint may also be effectively formulated as two bodies connected by a composite revolute-cylindrical joint. In the configuration shown in Fig. 7.12, the cylindrical-joint axis is perpendicular to the revolute-joint axis and the two axes intersect. Three vectors, $\vec{s}_i$, $\vec{s}_j$, and $\vec{d}$, can be defined for the constraint formulation. Vector $\vec{s}_i$ is located on the revolute-joint axis on body i, vector $\vec{s}_j$ is on the cylindrical-joint axis on body j, and vec-

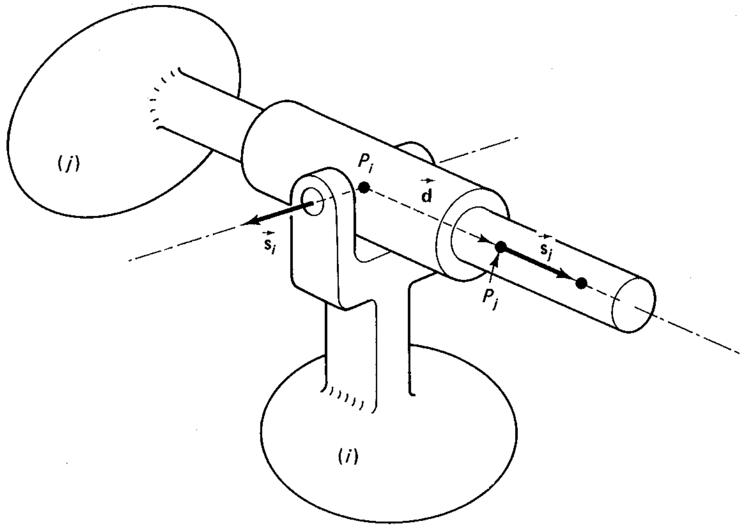


Figure 7.12 A revolute-cylindrical joint.

tor $\vec{d}$ connects point P_i on body i to point P_j on body j. The constraint equations are written as follows:

$$\begin{aligned} \Phi^{(n1,1)} &\equiv \mathbf{s}_i^T \mathbf{s}_j = 0 \\ \Phi^{(p2,2)} &\equiv \tilde{\mathbf{s}}_j \mathbf{d} = \mathbf{0} \end{aligned} \tag{7.26}$$

The first equation requires that the two axes remain perpendicular. The remaining equation guarantees that the two axes intersect at point P_i.

Other composite joints, similar to those in the preceding examples, may be defined and formulated. The formulation of composite joints may be simplified if body-fixed coordinates are placed on the bodies in such a way that unit vectors on the coordinate axes are used instead of arbitrary vectors on the joint axes. This process is similar to the special cases shown in Secs. 7.2.1 and 7.2.2.

7.2.4 Simplified Constraints

Translational constraint equations on the global coordinates of a body can be formulated in a manner similar to the process used for planar motion in Section 4.2.7. Such constraints are

$$\Phi \equiv x_i - c_1 = 0 \tag{7.27}$$

$$\Phi \equiv y_i - c_2 = 0 \tag{7.28}$$

$$\Phi \equiv z_i - c_3 = 0 \tag{7.29}$$

$$\Phi \equiv x_i^P - c_4 = 0 \tag{7.30}$$

$$\Phi \equiv y_i^P - c_5 = 0 \tag{7.31}$$

$$\Phi \equiv z_i^P - c_6 = 0 \tag{7.32}$$

where c_1 through c_6 are constants. Equations 7.27 through 7.29 constrain the origin of the body-fixed coordinate system, and Eqs. 7.30 through 7.32 constrain a point P_i on body i, in both cases relative to the global axes. The constants c_1 through c_6 can be replaced by time-dependent quantities for driving constraints.

7.3 POSITION, VELOCITY, AND ACCELERATION ANALYSIS

The computational aspects of spatial kinematics are the same as those of planar kinematic analysis. For a mechanism with b bodies, the vector of coordinates is represented as $\mathbf{q} = [\mathbf{q}_1^T, \mathbf{q}_2^T, \ldots, \mathbf{q}_b^T]^T$. Vector $\mathbf{q}_i$ is expressed in the form $\mathbf{q}_i = [x, y, z, e_0, e_1, e_2, e_3]_i^T$ or $\mathbf{q}_i = [\mathbf{r}^T, \mathbf{p}^T]_i^T$. If any artificial variables are used in the constraint formulation (for example, for screw joints), the artificial variables are also included in vector $\mathbf{q}$. The constraint equations are assumed to consist of m equations in the form

$$\mathbf{\Phi} \equiv \mathbf{\Phi}(\mathbf{q}) = \mathit{0} \tag{7.33}$$

Equation 7.33 contains $m - b$ kinematic constraints and b mathematical constraints between the Euler parameters (one equation per body) in the form given by Eq. 6.23.

Kinematic analysis with the appended driving constraint method requires a formulation identical to that shown in Sec. 3.2.2. For kinematic analysis, Jacobian matrix $\mathbf{\Phi}_\mathbf{q}$ must be evaluated.

Example 7.1

Determine the nonzero entries of the Jacobian matrix for constraint $\Phi^{(n1,1)}$ of Eq. 7.3.

Solution The nonzero entries are determined by evaluating the partial derivatives of Eq. 7.3 with respect to the coordinates of bodies i and j. Since $\mathbf{s}_i^T\mathbf{s}_j$ is not a function of $\mathbf{r}_i$, its partial derivative with respect to $\mathbf{r}_i$ is zero:

$$\Phi_{\mathbf{r}_i}^{(n1,1)} = \mathit{0}^T \tag{1}$$

The partial derivative of Eq. 7.3 can be found by writing the equation as $\mathbf{s}_j^T\mathbf{s}_i$ or $\mathbf{s}_j^T\mathbf{A}_i\mathbf{s}_i'$ and employing Eq. 6.86 to obtain

$$\Phi_{\mathbf{p}_i}^{(n1,1)} = 2\mathbf{s}_j^T(\mathbf{G}_i\bar{\mathbf{s}}_i' + \mathbf{s}_i'\mathbf{p}_i^T) \tag{2}$$

Similarly, the partial derivatives of Eq. 7.3 with respect to $\mathbf{r}_j$ and $\mathbf{p}_j$, by keeping the equation in the form $\mathbf{s}_i^T\mathbf{A}_j\mathbf{s}_j'$, are found to be

$$\Phi_{\mathbf{r}_j}^{(n1,1)} = \mathit{0}^T \tag{3}$$

$$\Phi_{\mathbf{p}_j}^{(n1,1)} = 2\mathbf{s}_i^T(\mathbf{G}_j\bar{\mathbf{s}}_j' + \mathbf{s}_j'\mathbf{p}_j^T) \tag{4}$$

Equations 2 and 4 each provide four nonzero entries in the columns of the Jacobian matrix associated with $\mathbf{p}_i$ and $\mathbf{p}_j$.

Table 7.1 summarizes elements of the Jacobian matrix for Eqs. 7.3 through 7.7 and Eq. 7.19. These equations are the essential building blocks for nearly all of the constraints developed in this chapter and for most constraint relations that can be developed between adjacent bodies. The fact that all terms can be written in compact vector or matrix form demonstrates an additional advantage of the Euler parameter formulation for developing compact computational algorithms.

TABLE 7.1 Components of the Jacobian Matrix of the Most Common Constraints[15]

Φ	$\Phi_{\mathbf{r}_i}$	$\Phi_{\mathbf{p}_i}$	$\Phi_{\mathbf{r}_j}$	$\Phi_{\mathbf{p}_j}$
$\Phi^{(n1,1)}$	0^T	$\mathbf{s}_j^T\mathbf{C}_i$	0^T	$\mathbf{s}_i^T\mathbf{C}_j$
$\Phi^{(n2,1)}$	$-\mathbf{s}_i^T$	$-\mathbf{s}_i^T\mathbf{B}_i + \mathbf{d}^T\mathbf{C}_i$	$\mathbf{s}_i^T$	$\mathbf{s}_i^T\mathbf{B}_j$
$\Phi^{(p1,2)}$	$\mathbf{0}$	$-\tilde{\mathbf{s}}_j\mathbf{C}_i$	$\mathbf{0}$	$\tilde{\mathbf{s}}_i\mathbf{C}_j$
$\Phi^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$-\tilde{\mathbf{s}}_i\mathbf{B}_i - \tilde{\mathbf{d}}\mathbf{C}_i$	$\tilde{\mathbf{s}}_i$	$\tilde{\mathbf{s}}_i\mathbf{B}_j$
$\Phi^{(s,3)}$	$\mathbf{I}$	$\mathbf{C}_i$	$-\mathbf{I}$	$-\mathbf{C}_j$
$\Phi^{(s\text{-}s,1)}$	$-2\mathbf{d}^T$	$-2\mathbf{d}^T\mathbf{B}_i$	$2\mathbf{d}^T$	$2\mathbf{d}^T\mathbf{B}_j$

where:

$$\mathbf{B}_k = 2(\mathbf{G}_k\bar{\mathbf{s}}_k'^B + \mathbf{s}_k'^B\mathbf{p}_k^T), \mathbf{C}_k = 2(\mathbf{G}_k\bar{\mathbf{s}}_k' + \mathbf{s}_k'\mathbf{p}_k^T), k = i,j$$

7.3.1 Modified Jacobian Matrix and Modified Vector γ

For most common constraints, the components of the Jacobian matrix of Table 7.1 can be employed directly. However, when the velocity and acceleration equations are considered, some equivalent terms from both sides of the equations can be eliminated. Hence, the final forms of the resultant equations are simpler, having fewer terms than the original equations.

Example 7.2

Consider a vector $\vec{s}_i$ whose magnitude and direction must remain fixed in a given problem. The constraints describing this condition can be expressed as

$$\begin{aligned}\Phi &\equiv \mathbf{s}_i - \mathbf{c}\\ &= \mathbf{A}_i\mathbf{s}_i' - \mathbf{c} = \mathit{0}\end{aligned} \tag{1}$$

where $\mathbf{c}$ contains three constant components. The entries of the Jacobian matrix for Eq. 1 are found, from Eq. 6.86, to be

$$\Phi_{\mathbf{p}_i} = 2\mathbf{G}_i\bar{\mathbf{s}}_i' + 2\mathbf{s}_i'\mathbf{p}_i^T \tag{2}$$

According to the first equation of Eq. 3.13, the velocity equation is

$$(2\mathbf{G}_i\bar{\mathbf{s}}_i' + 2\mathbf{s}_i'\mathbf{p}_i^T)\dot{\mathbf{p}}_i = \mathit{0} \tag{3}$$

Using the identity $\mathbf{p}_i^T\dot{\mathbf{p}}_i = \mathit{0}$ in Eq. 3 yields

$$2\mathbf{G}_i\bar{\mathbf{s}}_i'\dot{\mathbf{p}}_i = \mathit{0} \tag{4}$$

Similarly, according to the first equation of Eq. 3.15, the acceleration equation is

$$\begin{aligned}(2\mathbf{G}_i\bar{\mathbf{s}}_i' + 2\mathbf{s}_i'\mathbf{p}_i^T)\ddot{\mathbf{p}}_i &= -[(2\mathbf{G}_i\bar{\mathbf{s}}_i' + 2\mathbf{s}_i'\mathbf{p}_i^T)\dot{\mathbf{p}}_i]_{\mathbf{p}_i}\dot{\mathbf{p}}_i\\ &= (-2\dot{\mathbf{G}}_i\bar{\mathbf{s}}_i'\mathbf{p}_i - 2\mathbf{s}_i'\dot{\mathbf{p}}_i^T\mathbf{p}_i)_{\mathbf{p}_i}\dot{\mathbf{p}}_i\\ &= -2\dot{\mathbf{G}}_i\bar{\mathbf{s}}_i'\dot{\mathbf{p}}_i - 2\mathbf{s}_i'\dot{\mathbf{p}}_i^T\dot{\mathbf{p}}_i\end{aligned} \tag{5}$$

where Eqs. 6.73, 6.54, and 6.69 have been employed, in that order. From Eq. 6.61, i.e., $\mathbf{p}_i^T\ddot{\mathbf{p}}_i + \dot{\mathbf{p}}_i^T\dot{\mathbf{p}}_i = 0$, Eq. 5 becomes

$$\begin{aligned}2\mathbf{G}_i\bar{\mathbf{s}}_i'\ddot{\mathbf{p}}_i &= -2\dot{\mathbf{G}}_i\bar{\mathbf{s}}_i'\dot{\mathbf{p}}_i\\ &= -(2\mathbf{G}_i\bar{\mathbf{s}}_i'\dot{\mathbf{p}}_i)_{\mathbf{p}_i}\dot{\mathbf{p}}_i\end{aligned} \tag{6}$$

For velocity and acceleration analysis, Eqs. 4 and 6 can be used instead of Eqs. 3 and 5. By defining a modified Jacobian matrix as

$$\mathbf{\Phi}_{\mathbf{p}_i}^{(m)} = 2\mathbf{G}_i\bar{\mathbf{s}}_i' \tag{7}$$

we can write Eqs. 4 and 6 as

$$\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}\dot{\mathbf{p}}_i = \mathbf{0} \tag{8}$$

and

$$\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}\ddot{\mathbf{p}}_i = -(\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}\dot{\mathbf{p}}_i)_{\mathbf{p}_i}\dot{\mathbf{p}}_i \tag{9}$$

The simplification of the velocity and acceleration equations shown in Example 7.2 can be applied to all of the common constraint equations; it enables the entries of the modified Jacobian matrix and the components of the vector γ to be found. Another method for finding the modified Jacobian and vector γ is to use the second time derivative of the constraint equations and employ Eq. 6.82.

Example 7.3

Apply Eq. 6.82 to find the second time derivative of Eq. 7.3. Obtain the modified Jacobian matrix and modified vector γ for this constraint.

Solution The second time derivative of Eq. 7.3, i.e., $\mathbf{s}_i^T\mathbf{s}_j = 0$, is

$$\mathbf{s}_j^T\ddot{\mathbf{s}}_i + \mathbf{s}_i^T\ddot{\mathbf{s}}_j + 2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j = 0$$

or

$$\mathbf{s}_j^T\ddot{\mathbf{A}}_i\mathbf{s}_i' + \mathbf{s}_i^T\ddot{\mathbf{A}}_j\mathbf{s}_j' + 2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j = 0$$

With Eq. 6.82, the above equation yields

$$2\mathbf{s}_j^T(\dot{\mathbf{G}}_i\dot{\mathbf{L}}_i^T\mathbf{s}_i' + \mathbf{G}_i\bar{\mathbf{s}}_i'\ddot{\mathbf{p}}_i) + 2\mathbf{s}_i^T(\dot{\mathbf{G}}_j\dot{\mathbf{L}}_j^T\mathbf{s}_j' + \mathbf{G}_j\bar{\mathbf{s}}_j'\ddot{\mathbf{p}}_j) + 2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j = 0$$

This equation can be rearranged as follows:

$$[2\mathbf{s}_j^T\mathbf{G}_i\bar{\mathbf{s}}_i', 2\mathbf{s}_i^T\mathbf{G}_j\bar{\mathbf{s}}_j']\begin{bmatrix}\ddot{\mathbf{p}}_i\\ \ddot{\mathbf{p}}_j\end{bmatrix} = -2(\mathbf{s}_j^T\dot{\mathbf{G}}_i\dot{\mathbf{L}}_i\mathbf{s}_j' + \mathbf{s}_i^T\dot{\mathbf{G}}_j\dot{\mathbf{L}}_j\mathbf{s}_j' + \dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j)$$

The matrix on the left side of this equation shows the nonzero entries of the modified Jacobian, and the expression on the right side of this equation is the modified γ vector.

Table 7.2 summarizes the elements of the modified Jacobian matrix and modified vector γ for Eqs. 7.3 through 7.7 and Eq. 7.19. The components of Table 7.2 can be used in a computer program for kinematic analysis instead of those of Table 7.1. Examples 7.2 and 7.3 show how the entries of the Jacobian matrix can be modified (simplified) for velocity and acceleration analysis. It should be noted that the modified Jacobian matrix can also be used for position analysis. The first step of the Newton-Raphson iteration is given by Eq. 3.42 as

$$\mathbf{\Phi}_{\mathbf{q}}\Delta\mathbf{q} = -\mathbf{\Phi} \tag{a}$$

TABLE 7.2 Components in the Expansion of the Most Common Constraints[15]

$\mathbf{\Phi}$	$\mathbf{\Phi}_{\mathbf{r}_i}^{(m)}$	$\mathbf{\Phi}_{\mathbf{p}_i}^{(m)}$	$\mathbf{\Phi}_{\mathbf{r}_j}^{(m)}$	$\mathbf{\Phi}_{\mathbf{p}_j}^{(m)}$	$\gamma^{(m)}$
$\Phi^{(n1,1)}$	$\mathit{0}^T$	$2\mathbf{s}_j^T\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathit{0}^T$	$2\mathbf{s}_i^T\mathbf{G}_j\bar{\mathbf{s}}_j'$	$\mathbf{s}_i^T\mathbf{h}_j + \mathbf{s}_j^T\mathbf{h}_i - 2\dot{\mathbf{s}}_i^T\dot{\mathbf{s}}_j$
$\Phi^{(n2,1)}$	$-\mathbf{s}_i^T$	$-2\mathbf{s}_i^T\mathbf{G}_i\bar{\mathbf{s}}_i'^B + 2\mathbf{d}^T\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathbf{s}_i^T$	$2\mathbf{s}_i^T\mathbf{G}_j\mathbf{s}_j'^B$	$-\mathbf{s}_i^T(\mathbf{h}_i^B - \mathbf{h}_j^B) + \mathbf{d}^T\mathbf{h}_i - 2\dot{\mathbf{s}}_i^T\dot{\mathbf{d}}$
$\mathbf{\Phi}^{(p1,2)}$	$\mathbf{0}$	$-2\tilde{\mathbf{s}}_j\mathbf{G}_i\bar{\mathbf{s}}_i'$	$\mathbf{0}$	$2\tilde{\mathbf{s}}_i\mathbf{G}_j\bar{\mathbf{s}}_j'$	$\tilde{\mathbf{s}}_i\mathbf{h}_j - \tilde{\mathbf{s}}_j\mathbf{h}_i - 2\dot{\tilde{\mathbf{s}}}_i\dot{\mathbf{s}}_j$
$\mathbf{\Phi}^{(p2,2)}$	$-\tilde{\mathbf{s}}_i$	$-2\tilde{\mathbf{s}}_i\mathbf{G}_i\bar{\mathbf{s}}_i'^B - 2\tilde{\mathbf{d}}\mathbf{G}_i s_i'$	$\tilde{\mathbf{s}}_i$	$2\tilde{\mathbf{s}}_i\mathbf{G}_j\bar{\mathbf{s}}_j'^B$	$-\tilde{\mathbf{s}}_i(\mathbf{h}_j^B - \mathbf{h}_i^B) - \tilde{\mathbf{d}}\mathbf{h}_i - 2\dot{\tilde{\mathbf{s}}}_i\dot{\mathbf{d}}$
$\mathbf{\Phi}^{(s,3)}$	$\mathbf{I}$	$2\mathbf{G}_i\bar{\mathbf{s}}_i'^P$	$-\mathbf{I}$	$-2\mathbf{G}_j\bar{\mathbf{s}}_j'^P$	$\mathbf{h}_i^P - \mathbf{h}_j^P$
$\Phi^{(s\text{-}s,1)}$	$-2\mathbf{d}^T$	$-4\mathbf{d}^T\mathbf{G}_i\bar{\mathbf{s}}_i'^P$	$2\mathbf{d}^T$	$4\mathbf{d}^T\mathbf{G}_j\bar{\mathbf{s}}_j'^P$	$2\mathbf{d}^T(\mathbf{h}_i^P - \mathbf{h}_j^P) - 2\dot{\mathbf{d}}^T\dot{\mathbf{d}}$

where:

$$\mathbf{h}_k = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k', \mathbf{h}_k^B = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k'^B, \mathbf{h}_k^P = -2\dot{\mathbf{G}}_k\dot{\mathbf{L}}_k^T\mathbf{s}_k'^P, k = i, j$$

This operation involves the product of the Jacobian matrix and a vector $\Delta\mathbf{q}$ which contains the infinitesimal changes in the coordinates. Since the modification on the Jacobian has affected only the columns associated with the Euler parameters, not those for the translational coordinates, we need only test the part corresponding to $\mathbf{\Phi}_\mathbf{q}\,\Delta\mathbf{p}$. The first variation of the constraint equation on the Euler parameters (Eq. 6.23) is

$$2\mathbf{p}^T\Delta\mathbf{p} = 0 \qquad (b)$$

If we consider the entries of the Jacobian from Table 7.1 in the columns associated with $\mathbf{p}_i$ and $\mathbf{p}_j$ and employ Eq. *b*, we will obtain the modified matrix. For example, the columns of the Jacobian associated with $\mathbf{p}_i$ of $\Phi^{(n1,1)}$ yield

$$\begin{aligned}\Phi_{\mathbf{p}_i} &= (\mathbf{s}_j^T\mathbf{C}_i)\,\Delta\mathbf{p}_i \\ &= 2\mathbf{s}_j^T(\mathbf{G}_i\bar{\mathbf{s}}_i' + \mathbf{s}_i'\mathbf{p}_i^T)\,\Delta\mathbf{p}_i \\ &= 2\mathbf{s}_j^T\mathbf{G}_i\bar{\mathbf{s}}_i'\,\Delta\mathbf{p}_i \\ &= \Phi_{\mathbf{p}_i}^{(m)}\,\Delta\mathbf{p}_i\end{aligned}$$

This result can also be obtained by considering Eq. 6.86. This equation can be simplified in the Newton-Raphson algorithm as follows:

$$\begin{aligned}\left[\frac{\partial}{\partial\mathbf{p}}(\mathbf{A}\mathbf{a})\right]\Delta\mathbf{p} &= (2\mathbf{G}\bar{\mathbf{a}} + 2\mathbf{a}\mathbf{p}^T)\,\Delta\mathbf{p} \\ &= 2\mathbf{G}\bar{\mathbf{a}}\,\Delta\mathbf{p}\end{aligned}$$

This equation shows that the term $2\mathbf{G}\bar{\mathbf{a}}$ in Eq. 6.86 (or the term $2\mathbf{L}\overset{+}{\mathbf{a}}$ in Eq. 6.87) yields the entries of the modified Jacobian matrix. Therefore, two more identities can be stated:

$$\left[\frac{\partial}{\partial\mathbf{p}}(\mathbf{A}\mathbf{a})\right]^{(m)} = 2\mathbf{G}\bar{\mathbf{a}} \qquad (7.34)$$

and

$$\left[\frac{\partial}{\partial\mathbf{p}}(\mathbf{A}\mathbf{a})\right]^{(m)} = 2\mathbf{L}\overset{+}{\mathbf{a}} \qquad (7.35)$$

PROBLEMS

7.1 Two points B and C are located on body i. The coordinates of these points with respect to a set of $\xi_i\eta_i\zeta_i$ axes are $[1.2,\ 0.5,\ -0.1]^T$ and $[-0.3,\ -0.8, 2.1]^T$, respectively. The Euler parameters describing the angular orientation of body i with respect to a global coordinate system are $\mathbf{p}_i = [0.860, -0.150, 0.420, 0.248]^T$.

(a) Find the local components of vector $\vec{s} \equiv \overrightarrow{CB}$.

(b) Find the global components of vector $\vec{s}$.

(c) Is it possible to find the global coordinates of points B and C? Explain.

7.2 Point D is located on body i with local coordinates $[1.5, -1.6, 0.2]^T$. The origin of the $\xi_i\eta_i\zeta_i$ coordinate system is located with respect to the global coordinates by vector $\mathbf{r}_i = [3.3,\ 1.4,\ 2.0]^T$. The Euler parameters describing the angular orientation of body i are $\mathbf{p}_i = [0.810, -0.029, -0.543, 0.220]^T$. Determine the following:

(a) Local components of vector $\vec{s}_i^D$ connecting the origin of the $\xi_i\eta_i\zeta_i$ axes to point D

(b) Global components of vector $\vec{s}_i^D$

(c) Global coordinates of point D

7.3 Point B is located on body i with local coordinates $[1.0, 1.0, -0.5]^T$, and point C is located on body j with local coordinates $[-2.0, 1.5, -1.0]^T$. The origins of the bodies are located by vectors $\mathbf{r}_i = [-1.2, 0.4, 3.1]^T$ and $\mathbf{r}_j = [0.4, 4.5, 0.5]^T$ with respect to the global axes. The Euler parameters for the two bodies are $\mathbf{p}_i = [0.343, -0.564, 0.604, 0.447]^T$ and $\mathbf{p}_j = [0.270, 0.732, -0.331, 0.531]^T$. Find the following:

(a) Global coordinates of points B_i and C_j

(b) Global components of vector $\vec{d} \equiv \overrightarrow{B_iC_j}$

(c) Local components of vector $\vec{d}$ with respect to the $\xi_i\eta_i\zeta_i$ axes

(d) Local components of vector $\vec{d}$ with respect to the $\xi_j\eta_j\zeta_j$ axes

7.4 The orientations of bodies i and j are defined by Euler parameters $\mathbf{p}_i = [-0.667, -0.427, 0.241, -0.561]^T$ and, $\mathbf{p}_j = [0.223, 0.549, -0.623, 0.511]^T$. Vectors $\vec{s}_i$ and $\vec{s}_j$ are fixed vectors on the bodies with local components $\mathbf{s}_i' = [1, 1, 2]^T$ and $\mathbf{s}_j' = [-1, 1, \alpha]^T$, where α is unknown. Determine α, knowing that $\vec{s}_i$ and $\vec{s}_j$ are perpendicular.

7.5 What must the condition be between Euler parameters $\mathbf{p}_i$ and $\mathbf{p}_j$ for the two vectors $\vec{s}_i$ and $\vec{s}_j$ to remain perpendicular? Assume $\mathbf{s}_i' = [1, 1, 2]^T$ and $\mathbf{s}_j' = [2, -1, -2]^T$.

7.6 Points B and C are located on body i with local coordinates $[-0.2, -0.7, 0.6]^T$ and $[2.7, -0.7, -0.1]^T$. Vector $\vec{s}_i$ connects points B and C. Find a condition on the Euler parameters of this body for which vector $\vec{s}_i$ remains parallel to the xy plane.

7.7 Show that the three equations in $\tilde{\mathbf{s}}_i\mathbf{s}_j = \mathbf{0}$ are not independent.

7.8 Vectors $\mathbf{r}_i = [0.5, 0.5, 0.9]^T$ and $\mathbf{r}_j = [0.5, -0.2, 1.3]^T$ locate the fixed origins of bodies i and j with respect to the global axes. Vector $\vec{s}_i^C$ locates point C on body i as $\mathbf{s}_i'^C = [-0.7, 1.6, 0.8]^T$. Point B is located on body j by vector $\mathbf{s}_j'^B = [0.8, -0.6, -0.5]^T$. Write the constraint equation that makes vector $\vec{s}_i^C$ perpendicular to vector $\vec{d} \equiv \overrightarrow{C_iB_j}$.

7.9 Repeat Prob. 7.8 but assume that the origins of bodies i and j are not fixed; i.e., that vectors $\mathbf{r}_i$ and $\mathbf{r}_j$ are variables.

7.10 Write the condition for the two vectors $\vec{s}_i$ and $\vec{s}_j$ to be parallel. If the condition yields more than two equations, then choose the best two equations as constraint equations:

(a) $\mathbf{s}_i' = [-0.448, 0.399, 1.700]^T$, $\mathbf{p}_i = [0.860, -0.150, 0.420, 0.248]^T$,
$\mathbf{s}_j' = [-0.4131, 1.690, 0.634]^T$, $\mathbf{p}_j = [0.810, 0.029, -0.543, 0.220]^T$.

(b) $\mathbf{s}_i' = [1.976, 0.874, 1.825]^T$, $\mathbf{p}_i = [0.270, -0.732, 0.331, -0.531]^T$,
$\mathbf{s}_j' = [-1.143, -0.295, -1.220]^T$, $\mathbf{p}_j = [0.564, -0.447, -0.343, 0.604]^T$.

(c) $\mathbf{s}_i' = [-0.132, -0.089, -0.324]^T$, $\mathbf{p}_i = [-0.667, -0.427, 0.241, -0.561]^T$,
$\mathbf{s}_j' = [-0.446, 0.121, -0.082]^T$, $\mathbf{p}_j = [0.223, 0.549, -0.623, 0.511]^T$.

(d) $\mathbf{s}_i' = [1.64, 3, -1.52]^T$, $\mathbf{p}_i = [0.6, 0, -0.8, 0]^T$,
$\mathbf{s}_j' = [4.4, -6, 0.8]^T$, $\mathbf{p}_j = [0, 0.8, 0, 0.6]^T$.

7.11 What is (are) the necessary and sufficient condition(s) on the Euler parameters for the $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ coordinate systems to remain parallel?

7.12 For the $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ coordinate systems, what is (are) the necessary and sufficient condition(s) on the Euler parameters for the following conditions to be true:

(a) The ξ_i and ξ_j axes remain parallel.

(b) The ζ_i and η_j axes remain parallel.

(c) The η_i and η_j axes remain perpendicular.

7.13 Find the constraint equation for point P on body i to remain on the plane $z = 6$. The local coordinates of P are $\mathbf{s}_i'^P = [2, -2, -1]^T$.

7.14 Repeat Prob. 7.13 and constrain point P to move on the line described by the intersection of the two planes $z = 6$ and $x = 2$.

7.15 Figure P.7.15 shows a mechanism consisting of two bodies connected by a spherical joint. The vectors locating the center of the joint have components $\mathbf{s}_i'^P = [0.4, 1.4, -0.5]^T$ and $\mathbf{s}_j'^P = [0.9, -0.3, -0.2]^T$. Write the constraint equations for this joint. If the two bodies are not connected to any other bodies, how many degrees of freedom does the system have?

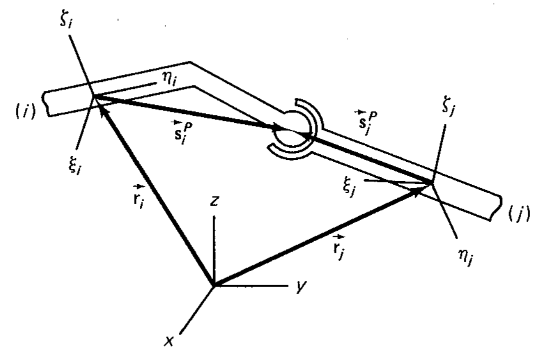


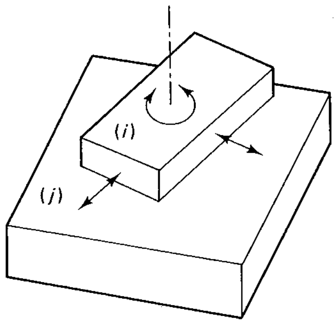
Figure P. 7.15

7.16 If two points P and Q are chosen arbitrarily on a revolute-joint axis connecting two bodies i and j, Eq. 7.7 may be repeated twice—for P and for Q—to yield six constraint equations. These equations can be used instead of Eq. 7.10. However, since there is one relative degree of freedom between the two bodies, one of the equations must be redundant. Define a strategy to eliminate one of the six equations efficiently in every possible configuration.

7.17 Two bodies are connected by a revolute joint. Find the constraint equation for this joint, using the spherical joint constraint and scalar products, instead of the vector product of Eq. 7.10. *Hint:* Initially define two vectors on one of the bodies perpendicular to the joint axis.

7.18 Add one equation to the cylindrical-joint constraints of Eq. 7.13 to allow them to be used as revolute-joint constraint equations.

7.19 The two bodies shown in Fig. P.7.19 can slide relative to each other without separation. Formulate constraint equations describing this type of joint.



7.20 Use the concept of relative axis of rotation (relative set of Euler parameters) to find an additional equation to convert cylindrical-joint constraints to translational-joint constraints.

7.21 Consider the system shown in Fig. 7.9, which contains two spherical joints. Determine the number of degrees of freedom if:

(a) The system is modeled by three bodies and two spherical joints.

(b) The system is modeled by two bodies and one spherical-spherical joint.

Compare the results from (a) and (b). Explain why they are different.

7.22 An A-arm suspension system contains a link connecting the main chassis (body i) to the wheel (body j) by two revolute joints as shown in Fig. P.7.22. The two joint axes intersect at an angle $\theta = 90°$. Determine a set of constraint equations to model this composite revolute-revolute joint.

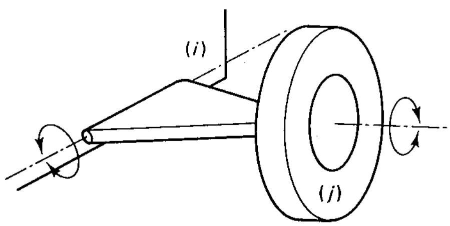


Figure P. 7.22

7.23 Repeat Prob. 7.22 for $\theta \neq 90°$.

7.24 The steering command in automobile simulations can be provided as a time-dependent constraint equation. This is usually done when the simulation of the actual steering mechanism is not of interest. Assume that body i is the main chassis, with η_i along the longitudinal direction and body j as one of the front wheels. In Figure P.7.24, the suspension mechanism between the wheel and the chassis is not shown. If the steering command is described as $c(t)$, derive a constraint equation between two unit vectors along the η_i and η_j axes.

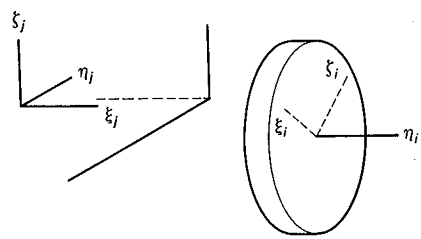


Figure P. 7.24

7.25 Verify the entries of the Jacobian matrix listed in Table 7.1.

7.26 Derive expressions for vector γ for the constraint equations that are listed in Table 7.1

7.27 Verify the entries of the modified Jacobian matrix and modified vector γ listed in Table 7.2.

7.28 The road wheels of a tracked vehicle are connected to the chassis by road arms as shown in Fig. P.7.28. A road arm can be modeled as a composite revolute-revolute joint with parallel axes. The constraint equations for this composite joint may be simplified by locating the local coordinate systems on the chassis and on the wheels so that they have parallel axes; e.g., the ξ axes could be parallel as shown coming out of the plane. Derive the simplified constraint equations.

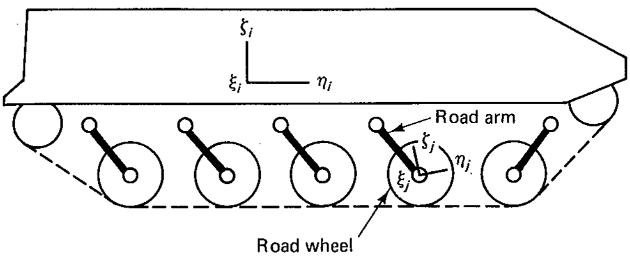


Figure P. 7.28