6

Euler Parameters

In this chapter and the next, spatial kinematics is discussed. Although the analytical procedure in spatial kinematics is the same as in the planar case, spatial kinematic analysis requires much more powerful mathematical techniques than planar kinematics, particularly for describing the angular orientation of a body in a global coordinate system. Therefore, this chapter is mostly devoted to developing the techniques involved in describing the angular orientation of bodies in space, without being concerned with the translation.

As its title suggests, this chapter concentrates on a set of orientational coordinates known as *Euler parameters,*† which are free of some of the deficiencies of other commonly used angular coordinates, such as Euler angles. At the beginning, it may appear that Euler parameters have no physical significance and that they are just mathematical tools. However, when the subject is thoroughly understood, their physical relevance will also become evident. Furthermore, for large-scale computer programs that treat the angular orientation of bodies, either rigid or deformable, the use of Euler parameters may drastically simplify the mathematical formulations.

6.1 COORDINATES OF A BODY

An unconstrained body in space requires six independent coordinates to determine its configuration — three coordinates specify translation and three specify rotation. The six coordinates define the location of a Cartesian coordinate system that is fixed in the body (i.e., the location of its local, or body-fixed, coordinates) relative to the global (refer-

†Euler parameters are a normalized form of parameters known as *quaternions.*[19]

ence or inertial) coordinate axes. Since all points in the body may be located relative to this body-fixed coordinate system, the global locations of all points in the body can thus be determined from the six coordinates. The coordinates of the origin of the body-fixed axes are the translational coordinates. Rotational coordinates are then needed to define the orientation of the local axes relative to the global coordinate axes. Throughout this text, the body-fixed axes will be denoted as $\xi\eta\zeta$ axes and the global axes will be denoted as xyz axes.

Figure 6.1(a) shows how the configuration of the $\xi\eta\zeta$ axes with respect to the xyz axes can be considered a translation (xyz to $x'y'z'$) and a rotation ($x'y'z'$ to $\xi\eta\zeta$). However, for purposes of finding only the angular orientation of the $\xi\eta\zeta$ axes relative to the xyz system, the origins of the two systems may be considered to coincide, as shown in Fig. 6.1(b).

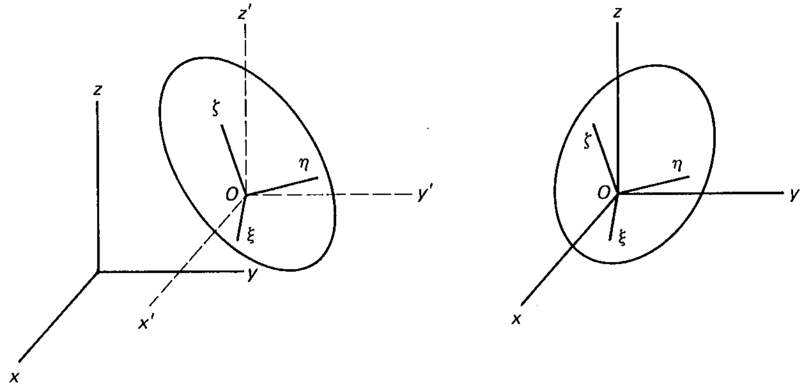

Figure 6.1 Configuration of Cartesian coordinate systems: (a) translation and rotation; (b) rotation only.

A vector $\vec{s}$ from the origin to a point P, as shown in Fig. 6.2, can be expanded in either of the two coordinate systems. If unit vectors $\vec{u}_{(\xi)}$, $\vec{u}_{(\eta)}$, and $\vec{u}_{(\zeta)}$ are defined along the $\xi\eta\zeta$ axes and $\vec{u}_{(x)}$, $\vec{u}_{(y)}$, and $\vec{u}_{(z)}$ are defined along the xyz axes, then:

$$\vec{s} = s_{(x)}\vec{u}_{(x)} + s_{(y)}\vec{u}_{(y)} + s_{(z)}\vec{u}_{(z)} \tag{6.1}$$

or

$$\vec{s} = s_{(\xi)}\vec{u}_{(\xi)} + s_{(\eta)}\vec{u}_{(\eta)} + s_{(\zeta)}\vec{u}_{(\zeta)} \tag{6.2}$$

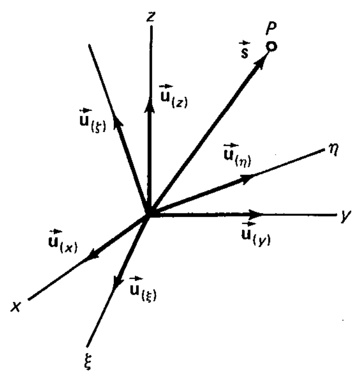

Figure 6.2 Unit vectors along the axes of the local and global coordinate systems.

where

$$s_{(x)} = \vec{s} \cdot \vec{u}_{(x)}, \qquad s_{(y)} = \vec{s} \cdot \vec{u}_{(y)}, \qquad s_{(z)} = \vec{s} \cdot \vec{u}_{(z)}$$

and

$$s_{(\xi)} = \vec{s} \cdot \vec{u}_{(\xi)}, \qquad s_{(\eta)} = \vec{s} \cdot \vec{u}_{(\eta)}, \qquad s_{(\zeta)} = \vec{s} \cdot \vec{u}_{(\zeta)}$$

The component vectors that define $\vec{s}$ in the two coordinate systems are

$$\mathbf{s} = [s_{(x)}, s_{(y)}, s_{(z)}]^T$$

in the xyz system and

$$\mathbf{s}' = [s_{(\xi)}, s_{(\eta)}, s_{(\zeta)}]^T$$

in the $\xi\eta\zeta$ system. It is clear that there is a relation between $\mathbf{s}$ and $\mathbf{s}'$, since they are uniquely defined by the same vector $\vec{s}$. To find this relation, the $\vec{u}_{(\xi)}$, $\vec{u}_{(\eta)}$, and $\vec{u}_{(\zeta)}$ unit vectors are defined in terms of the $\vec{u}_{(x)}$, $\vec{u}_{(y)}$, and $(\vec{u})_{(z)}$ unit vectors as follows:

$$\begin{aligned}
\vec{u}_{(\xi)} &= a_{11}\vec{u}_{(x)} + a_{21}\vec{u}_{(y)} + a_{31}\vec{u}_{(z)} \\
\vec{u}_{(\eta)} &= a_{12}\vec{u}_{(x)} + a_{22}\vec{u}_{(y)} + a_{32}\vec{u}_{(z)} \\
\vec{u}_{(\zeta)} &= a_{13}\vec{u}_{(x)} + a_{23}\vec{u}_{(y)} + a_{33}\vec{u}_{(z)}
\end{aligned} \tag{6.3}$$

where $a_{ij}, i, j = 1, 2, 3$, are the direction cosines that can be expressed as

$$\begin{aligned}
a_{11} &= \vec{u}_{(\xi)} \cdot \vec{u}_{(x)} = \cos(\vec{u}_{(\xi)}, \vec{u}_{(x)}) \\
a_{21} &= \vec{u}_{(\xi)} \cdot \vec{u}_{(y)} = \cos(\vec{u}_{(\xi)}, \vec{u}_{(y)}) \\
a_{31} &= \vec{u}_{(\xi)} \cdot \vec{u}_{(z)} = \cos(\vec{u}_{(\xi)}, \vec{u}_{(z)}) \\
a_{12} &= \vec{u}_{(\eta)} \cdot \vec{u}_{(x)} = \cos(\vec{u}_{(\eta)}, \vec{u}_{(x)}) \\
a_{22} &= \vec{u}_{(\eta)} \cdot \vec{u}_{(y)} = \cos(\vec{u}_{(\eta)}, \vec{u}_{(y)}) \\
a_{32} &= \vec{u}_{(\eta)} \cdot \vec{u}_{(z)} = \cos(\vec{u}_{(\eta)}, \vec{u}_{(z)}) \\
a_{13} &= \vec{u}_{(\zeta)} \cdot \vec{u}_{(x)} = \cos(\vec{u}_{(\zeta)}, \vec{u}_{(x)}) \\
a_{23} &= \vec{u}_{(\zeta)} \cdot \vec{u}_{(y)} = \cos(\vec{u}_{(\zeta)}, \vec{u}_{(y)}) \\
a_{33} &= \vec{u}_{(\zeta)} \cdot \vec{u}_{(z)} = \cos(\vec{u}_{(\zeta)}, \vec{u}_{(z)})
\end{aligned} \tag{6.4}$$

Substituting from Eq. 6.3 into Eq. 6.2 yields

$$\begin{aligned}
\vec{s} = {} & (a_{11}s_{(\xi)} + a_{12}s_{(\eta)} + a_{13}s_{(\zeta)})\vec{u}_{(x)} \\
& + (a_{21}s_{(\xi)} + a_{22}s_{(\eta)} + a_{23}s_{(\zeta)})\vec{u}_{(y)} \\
& + (a_{31}s_{(\xi)} + a_{32}s_{(\eta)} + a_{33}s_{(\zeta)})\vec{u}_{(z)}
\end{aligned} \tag{6.5}$$

By equating the right sides of Eqs. 6.1 and 6.5, it is found that

$$\begin{aligned}
s_{(x)} &= a_{11}s_{(\xi)} + a_{12}s_{(\eta)} + a_{13}s_{(\zeta)} \\
s_{(y)} &= a_{21}s_{(\xi)} + a_{22}s_{(\eta)} + a_{23}s_{(\zeta)} \\
s_{(z)} &= a_{31}s_{(\xi)} + a_{32}s_{(\eta)} + a_{33}s_{(\zeta)}
\end{aligned}$$

or, in matrix form,

$$\mathbf{s} = \mathbf{A}\mathbf{s}' \tag{6.6}$$

where the matrix $\mathbf{A}$ of direction cosines is

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{6.7}$$

The matrix **A** has a special property. If the xyz components of unit vectors $\vec{u}_{(\xi)}$, $\vec{u}_{(\eta)}$, and $\vec{u}_{(\zeta)}$ are denoted by $\mathbf{u}_{(\xi)}$, $\mathbf{u}_{(\eta)}$, and $\mathbf{u}_{(\zeta)}$ and the xyz components of vectors $\vec{u}_{(x)}$, $\vec{u}_{(y)}$, and $\vec{u}_{(z)}$ are denoted by $\mathbf{u}_{(x)}$, $\mathbf{u}_{(y)}$, and $\mathbf{u}_{(z)}$, it is clear that

$$\mathbf{u}_{(x)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_{(y)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_{(z)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.8}$$

Equation 6.4 indicates that a_{11} is the x component of $\mathbf{u}_{(\xi)}$, a_{21} is the y component of $\mathbf{u}_{(\xi)}$, and so forth. Therefore,

$$\mathbf{u}_{(\xi)} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \quad \mathbf{u}_{(\eta)} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \quad \mathbf{u}_{(\zeta)} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

and the matrix **A** can be written as follows:

$$\mathbf{A} = [\mathbf{u}_{(\xi)}, \mathbf{u}_{(\eta)}, \mathbf{u}_{(\zeta)}] \tag{6.9}$$

Since the unit vectors $\mathbf{u}_{(\xi)}$, $\mathbf{u}_{(\eta)}$, and $\mathbf{u}_{(\zeta)}$ are *orthogonal,*

$$\mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{6.10}$$

Thus, $\mathbf{A}^T = \mathbf{A}^{-1}$, and the matrix **A** is also *orthogonal.* This special property permits an easy inversion of Eq. 6.6, to obtain

$$\mathbf{s}' = \mathbf{A}^T\mathbf{s} \tag{6.11}$$

The nine direction cosines in **A** define the orientation of the $\xi\eta\zeta$ axes relative to the xyz axes, but they are not independent. Substituting Eq. 6.4 into Eq. 6.10 provides six equations (three of the nine equations are repeated twice) among the nine direction cosines. Thus, only three direction cosines are independent. While the nine direction cosines, subject to six constraints, could be adopted as rotational coordinates, this is neither practical nor convenient. Thus, other orientation coordinates are sought.

When the origins of the xyz and $\xi\eta\zeta$ coordinate systems do not coincide, as is the case in Fig. 6.1(a), the foregoing analysis is applied between the $x'y'z'$ and $\xi\eta\zeta$ systems. If the component vector $\mathbf{s}'^P$ locates a point P in the $\xi\eta\zeta$ coordinate system, as it does in Fig. 6.3, then in the $x'y'z'$ system this vector is just $\mathbf{A}\mathbf{s}'^P$, and in global xyz coordinates,

$$\mathbf{r}^P = \mathbf{r} + \mathbf{A}\mathbf{s}'^P \tag{6.12}$$

where $\mathbf{r}$ is the vector from the origin of the xyz system to the origin of the $\xi\eta\zeta$ system.

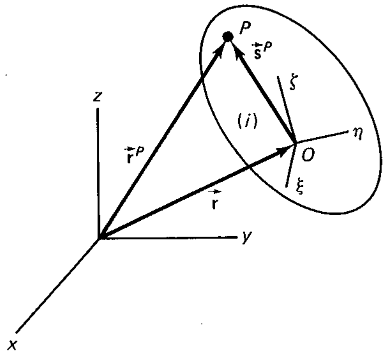

Figure 6.3 Translation and rotation of a body in three-dimensional space.

6.1.1 Euler's Theorem on the Motion of a Body

At any instant in time, the orientation of a body can be specified by a transformation matrix, the elements of which may be expressed in terms of suitable sets of coordinates. As time progresses, the orientation of the body will change. Hence the transformation matrix will be a function of time. Since the motion of the body is continuous, the transformation matrix must be a continuous function of time. The transformation may thus be said to *evolve continuously*. With this method of describing the motion, *Euler's theorem*[4] can be stated as follows:

> **Euler's theorem:** *The general displacement of a body with one point fixed is a rotation about some axis.*

The theorem indicates that the orientation of the body-fixed axes at any time t can be obtained by an imaginary rotation of these axes from an orientation coincident with the global axes. This imaginary axis of rotation *is not* the so-called instantaneous axis of rotation of the body—in this text we will call it the *orientational axis of rotation*. It is important to note that any vector lying along the orientational axis of rotation is left unaffected by this imaginary rotation—it must have the same components in both the reference and the body-fixed coordinates. The other necessary condition for rotation, that the magnitude of vectors undergoing the imaginary rotation be unaffected, is automatically satisfied.

An immediate corollary of Euler's theorem, known as *Chasles's theorem,*[4] is stated as follows:

> **Chasles's theorem:** *The most general displacement of a body is a translation plus a rotation.*

This theorem simply states that removing the constraint of motion with one point fixed introduces three translatory degrees of freedom for the origin of the body-fixed axes.

6.1.2 Active and Passive Points of View

A change in the angular orientation may be interpreted from an *active* point of view or from a *passive* point of view. Symbolically, a transformation may be written as

$$\mathbf{s} = \mathbf{A}\mathbf{s}' \tag{6.13}$$

According to the active point of view, the operator $\mathbf{A}$ relates two vectors of equal length, $\vec{s}$ and $\vec{s}'$, expressed in terms of the global coordinate system only, as shown in Fig. 6.4(a). On the other hand, the passive point of view describes only a single vector $\vec{s}$ and introduces a new local coordinate system to account for the change in orientation, as shown in Fig. 6.4(b). In this case the operator $\mathbf{A}$ relates the global components of the vector $\vec{s}$ to its local components; i.e., $\mathbf{s}$ and $\mathbf{s}'$. Whereas one rotates the coordinate system counterclockwise (positive sense of rotation), according to the passive point of view, one rotates vector $\vec{s}$ clockwise by the same angle from the active point of view, to obtain a new vector $\vec{s}'$ in the same coordinate system. The algebra is the same when either of the two points of view is followed.

In the following sections, rotational coordinates known as Euler parameters are discussed. The Euler parameter set employs the active point of view for determination of the transformation matrix $\mathbf{A}$. A discussion of two other sets of commonly used coordinates, known as Euler angles and Bryant angles, can be found in Appendix A. The pas-

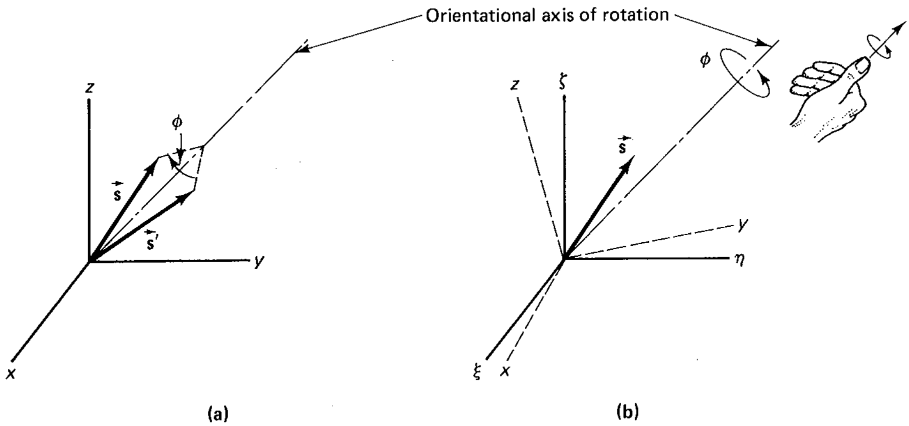

Figure 6.4 Coordinate system rotation: (a) active point of view; (b) passive point of view.

sive point of view is employed to determine the transformation matrix in terms of Euler and Bryant angles.

6.1.3 Euler Parameters

Euler's theorem states that a coordinate transformation can be accomplished by a single rotation about a suitable axis. It is natural, therefore, to seek a representation of the coordinate transformation in terms of parameters of this rotation, namely, the angle of rotation and the direction cosines of the orientational axis of rotation.[4]

In Fig. 6.5 the initial position of the vector $\vec{s}$ is denoted by $\overrightarrow{OP}$ and the final position $\vec{s}'$ is denoted by $\overrightarrow{OP'}$. The unit vector along the orientational axis of rotation is denoted by $\vec{u}$. Vector $\vec{s}$ can be expressed as the sum of three vectors:

$$\vec{s} = \overrightarrow{ON} + \overrightarrow{NQ} + \overrightarrow{QP} \qquad (a)$$

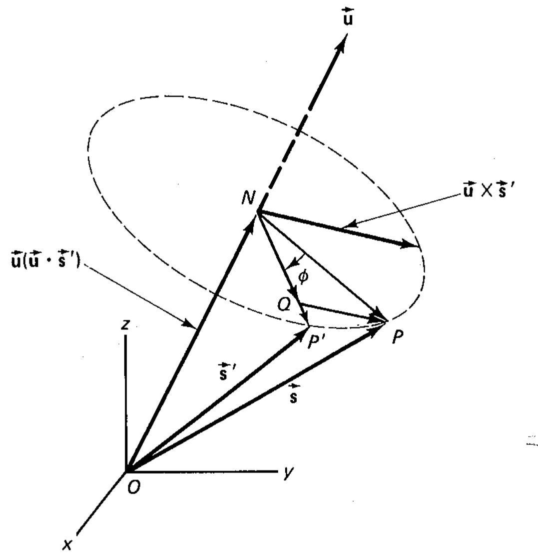

Figure 6.5 Vector diagram for derivation of rotation formula.

The direct distance between points O and N is $\vec{u} \cdot \vec{s}'$, so vector $\overrightarrow{ON}$ can be written as follows:

$$\overrightarrow{ON} = \vec{u}(\vec{u} \cdot \vec{s}') \tag{b}$$

Vector $\overrightarrow{NP'}$ can also be described as follows:

$$\overrightarrow{NP'} = \vec{s}' - \overrightarrow{ON} = \vec{s}' - \vec{u}(\vec{u} \cdot \vec{s}')$$

Hence,

$$\overrightarrow{NQ} = [\vec{s}' - \vec{u}(\vec{u} \cdot \vec{s}')] \cos \phi \tag{c}$$

The magnitude of vector $\overrightarrow{NP'}$ is the same as that of vectors $\overrightarrow{NP}$ and $\vec{u} \times \vec{s}'$. Therefore, vector $\overrightarrow{QP}$ may be expressed as

$$\overrightarrow{QP} = \vec{u} \times \vec{s}' \sin \phi \tag{d}$$

Substitution of Eqs. b, c, and d into Eq. a, together with a slight rearrangement of terms, leads to the *rotation formula*:[4]

$$\vec{s} = \vec{s}' \cos \phi + \vec{u}(\vec{u} \cdot \vec{s}')(1 - \cos \phi) + \vec{u} \times \vec{s}' \sin \phi \tag{6.14}$$

By means of the standard trigonometric relationships

$$\cos \phi = 2 \cos^2 \frac{\phi}{2} - 1$$

$$\sin \phi = 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2}$$

$$1 - \cos \phi = 2 \sin^2 \frac{\phi}{2}$$

and the new quantities

$$\begin{aligned} e_0 &= \cos \frac{\phi}{2} \\ \vec{e} &= \vec{u} \sin \frac{\phi}{2} \end{aligned} \tag{6.15}$$

the rotation formula of Eq. 6.14 can be put in a more useful form:

$$\vec{s} = (2e_0^2 - 1)\vec{s}' + 2\vec{e}(\vec{e} \cdot \vec{s}') + 2e_0\vec{e} \times \vec{s}' \tag{6.16}$$

Algebraic representation of Eq. 6.16, using the component form $\mathbf{e} = [e_1, e_2, e_3]^T$ of $\vec{e}$, yields

$$\mathbf{s} = (2e_0^2 - 1)\mathbf{s}' + 2\mathbf{e}(\mathbf{e}^T\mathbf{s}') + 2e_0(\tilde{\mathbf{e}}\mathbf{s}')$$

or

$$\mathbf{s} = [(2e_0^2 - 1)\mathbf{I} + 2\mathbf{e}\mathbf{e}^T + 2e_0\tilde{\mathbf{e}}]\mathbf{s}' \tag{6.17}$$

where $\mathbf{I}$ is the 3×3 identity matrix and, by the definition in Eq. 2.43,

$$\tilde{\mathbf{e}} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

The term in brackets in Eq. 6.17 is thus the transformation matrix of Eq. 6.13; i.e.,

$$\mathbf{A} = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}}) \tag{6.18}$$

More explicitly,

$$\mathbf{A} = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix} \tag{6.19}$$

Taking the transpose of both sides of Eq. 6.18 yields

$$\mathbf{A}^T = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T - e_0\tilde{\mathbf{e}}) \tag{6.20}$$

The four quantities e_0, e_1, e_2, and e_3 are called *Euler parameters*. Equation 6.15 indicates that the Euler parameters are not independent. Since $\cos^2(\phi/2) + \mathbf{u}^T\mathbf{u}\sin^2(\phi/2) = 1$, then

$$e_0^2 + \mathbf{e}^T\mathbf{e} = 1 \tag{6.21}$$

i.e.,

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

If the four Euler parameters are put in a 4-vector as follows:

$$\begin{aligned} \mathbf{p} &= [e_0, e^T]^T \\ &= [e_0, e_1, e_2, e_3]^T \end{aligned} \tag{6.22}$$

then Eq. 6.21 is written as

$$\mathbf{p}^T\mathbf{p} - 1 = 0 \tag{6.23}$$

According to Euler's theorem, any vector lying along the orientational axis of rotation must have the same components in both initial and final coordinate systems. This statement may be verified by finding the local and global components of the vector $\vec{e}$. Assume that $\mathbf{e} = [e_1, e_2, e_3]^T$ consists of the global components of $\vec{e}$. The transformation matrix $\mathbf{A}$ can be used to obtain the local components of $\vec{e}$; i.e., $\mathbf{e}'$, as follows:

$$\begin{aligned} \mathbf{e}' &= \mathbf{A}^T\mathbf{e} \\ &= (2e_0^2 - 1)\mathbf{e} + 2(\mathbf{e}\mathbf{e}^T - e_0\tilde{\mathbf{e}})\mathbf{e} \\ &= (2e_0^2 - 1)\mathbf{e} + 2\mathbf{e}(1 - e_0^2) \\ &= (2e_0^2 - 1)\mathbf{e} + 2(1 - e_0^2)\mathbf{e} \\ &= (2e_0^2 - 1 + 2 - 2e_0^2)\mathbf{e} \\ &= \mathbf{e} \end{aligned}$$

where Eqs. 6.18 and 6.21 and the identity $\tilde{\mathbf{e}}\mathbf{e} = \mathbf{0}$ (Eq. 2.48) have been used. This result shows that the global components and the local components of $\vec{e}$ are the same. Figure 6.6 illustrates the projection of $\vec{e}$ on both the $\xi\eta\zeta$ and the xyz axes.

6.1.4 Determination of Euler Parameters

From the transformation matrix of Eq. 6.19, it is possible to derive explicit formulas for the Euler parameters in terms of the elements of the transformation matrix. Assume that the nine direction cosines of a transformation matrix are given as in Eq. 6.7:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

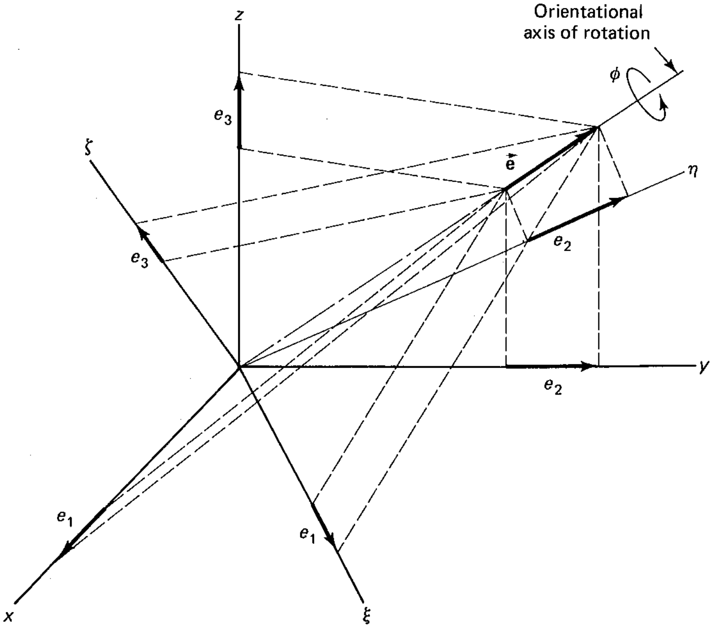

Figure 6.6 Projection of vector $\vec{e}$ on ξ, η, ζ, x, y, and z axes.

The trace of **A**, denoted by tr **A**, is defined as follows:

$$\text{tr } \mathbf{A} = a_{11} + a_{22} + a_{33} \tag{6.24}$$

From the transformation matrix of Eq. 6.19 it is found that

$$\begin{aligned} \text{tr } \mathbf{A} &= 2(3e_0^2 + e_1^2 + e_2^2 + e_3^2) - 3 \\ &= 2(2e_0^2 + 1) - 3 \\ &= 4e_0^2 - 1 \end{aligned}$$

where Eq. 6.21 has been employed. This equation can be written as

$$e_0^2 = \frac{\text{tr } \mathbf{A} + 1}{4} \tag{6.25}$$

Substituting this into the diagonal elements of **A** results in

$$\begin{aligned} a_{11} &= 2(e_0^2 + e_1^2) - 1 \\ &= 2\left(\frac{\text{tr } \mathbf{A} + 1}{4} + e_1^2\right) - 1 \end{aligned}$$

or

$$e_1^2 = \frac{1 + 2a_{11} - \text{tr } \mathbf{A}}{4} \tag{6.26a}$$

and similarly,

$$e_2^2 = \frac{1 + 2a_{22} - \text{tr } \mathbf{A}}{4} \tag{6.26b}$$

and

$$e_3^2 = \frac{1 + 2a_{33} - \text{tr } \mathbf{A}}{4} \tag{6.26c}$$

In contrast to Euler and Bryant angles (see Appendix A), or any other set of three rotational coordinates, there are no critical cases in which these inverse formulas are singular.

It is interesting and computationally important to note that Eqs. 6.25 and 6.26 determine only the magnitudes of the Euler parameters, in terms of only the diagonal elements of the direction-cosine matrix **A**. To find the algebraic signs of the Euler parameters, off-diagonal terms must be used. Equation 6.21 indicates that at least one Euler parameter must be nonzero, e.g., e_0. The sign of e_0 may be selected as positive or negative. Subtracting symmetrically placed off-diagonal terms of matrix **A** in Eqs. 6.7 and 6.19 yields

$$\begin{aligned} a_{32} - a_{23} &= 4e_0e_1 \\ a_{13} - a_{31} &= 4e_0e_2 \\ a_{21} - a_{12} &= 4e_0e_3 \end{aligned}$$

or

$$\begin{aligned} e_1 &= \frac{a_{32} - a_{23}}{4e_0} \\ e_2 &= \frac{a_{13} - a_{31}}{4e_0} \\ e_3 &= \frac{a_{21} - a_{12}}{4e_0} \end{aligned} \tag{6.27}$$

If e_0, calculated from Eq. 6.25, is nonzero, then Eq. 6.27 can be used to determine e_1, e_2, and e_3. Suppose that for an assumed sign for e_0, and for the computed values of e_1, e_2, and e_3, the angle of rotation and the axis of rotation are determined to be ϕ and $\vec{e}$, respectively. If the sign of e_0 is inverted, the signs of e_1, e_2, and e_3 are inverted also. Changing the signs of all four parameters does not influence the transformation matrix, since the matrix is quadratic.

If e_0, calculated from Eq. 6.25, is found to be zero, then Eqs. 6.26a–c can be used to calculate e_1, e_2, and e_3. Since $e_0 = 0$, Eq. 6.15 indicates that $\phi = k\pi$, $k = \pm 1, \pm 3, \ldots$. Therefore, the sign of ϕ is immaterial; e.g., $+\pi$ and $-\pi$ are the same. To find the algebraic sign of e_1, e_2, and e_3, symmetrically placed off-diagonal terms of matrix **A** are added to obtain

$$\begin{aligned} a_{21} + a_{12} &= 4e_1e_2 \\ a_{31} + a_{13} &= 4e_1e_3 \\ a_{32} + a_{23} &= 4e_2e_3 \end{aligned} \tag{6.28}$$

At least one of the three Euler parameters e_1, e_2, and e_3 must be nonzero. Its sign may be selected as positive or negative. Then, Eq. 6.28 can be used to determine the magnitude and the sign of the other two parameters.

Example 6.1

Nine direction cosines of a transformation matrix **A** are given as follows:

$$\mathbf{A} = \begin{bmatrix} 0.5449 & -0.5549 & 0.6285 \\ 0.3111 & 0.8299 & 0.4629 \\ -0.7785 & -0.0567 & 0.6249 \end{bmatrix}$$

Determine the four Euler parameters corresponding to this transformation.

Solution The trace of $\mathbf{A}$ is calculated from Eq. 6.24:

$$\text{tr } \mathbf{A} = 0.5449 + 0.8299 + 0.6249 = 1.9997$$

Then, Eq. 6.25 yields $e_0^2 = 0.7499$. Selecting the positive sign for e_0, we find that $e_0 = 0.866$. From Eq. 6.27,

$$e_1 = \frac{-0.0567 - 0.4629}{4.0(0.866)} = -0.15$$

$$e_2 = \frac{0.6285 + 0.7785}{4.0(0.866)} = 0.406$$

$$e_3 = \frac{0.3111 + 0.5549}{4.0(0.866)} = 0.25$$

A test can be performed to check that the four parameters satisfy the constraint of Eq. 6.21. Either the four parameters are $\mathbf{p} = [0.866, -0.15, 0.406, 0.25]^T$, or, if the sign of e_0 is changed, the four parameters become $\mathbf{p} = [-0.866, 0.15, -0.406, -0.25]^T$.

Example 6.2

Determine the four Euler parameters for transformation matrix

$$\mathbf{A} = \begin{bmatrix} -0.280 & -0.600 & -0.749 \\ -0.600 & -0.500 & 0.625 \\ -0.749 & 0.625 & -0.220 \end{bmatrix}$$

Solution The trace of $\mathbf{A}$ is found from Eq. 6.24:

$$\text{tr } \mathbf{A} = -0.280 - 0.500 - 0.220 = -1.0$$

Then, Eq. 6.25 yields $e_0 = 0.0$. From Eq. 6.26 it is found that

$$e_1^2 = \frac{1.0 + 2.0(-0.28) + 1.0}{4.0} = 0.36$$

Therefore, $e_1 = \pm 0.6$. If the positive sign is selected for e_1, then, Eq. 6.28 yields

$$e_2 = \frac{-0.6 - 0.6}{4.0(0.6)} = -0.5$$

$$e_3 = \frac{-0.749 - 0.749}{4.0(0.6)} = -0.624$$

The vector of the Euler parameters is $\mathbf{p} = [0.0, 0.6, -0.5, -0.624]^T$ or $\mathbf{p} = [0.0, -0.6, 0.5, 0.624]^T$.

When the angle of rotation is $\phi = k\pi$, $k = \pm 1, \pm 3, \ldots$, then e_0 is zero. Therefore, the transformation matrix of Eq. 6.19 becomes

$$\mathbf{A} = 2\begin{bmatrix} e_1^2 - \frac{1}{2} & e_1 e_2 & e_1 e_3 \\ e_1 e_2 & e_2^2 - \frac{1}{2} & e_2 e_3 \\ e_1 e_3 & e_2 e_3 & e_3^2 - \frac{1}{2} \end{bmatrix} \tag{6.29}$$

which is symmetric. This property was observed in Example 6.2.

6.1.5 Determination of the Direction Cosines

It was shown in Section 6.1.4 that the Euler parameters can be determined if the direction cosines are known. This section considers methods to determine the direction cosines.

One method for determining the nine direction cosines that describe the orientation of a body-fixed coordinate system with respect to the reference coordinates is to use Euler angles. If the three Euler angles can be determined (refer to Appendix A), then the elements of the transformation matrix can be computed. A direct calculation of the four Euler parameters in terms of the three Euler angles is given in Appendix B. This method may seem to be simple and straightforward; however, determination of the three Euler angles is difficult, and, for general cases, impractical.

A second method for determining the nine direction cosines is discussed here. Two points A and B are located on the ξ and η axes, as shown in Fig. 6.7. The xyz coordinates of A and B and the origin O can be found by measurements taken on the actual system or on an illustration, or by some other means. The coordinates of points O, A, and B are denoted by $\mathbf{r}$, $\mathbf{r}^A$, and $\mathbf{r}^B$, respectively. Vectors $\vec{a}$ and $\vec{b}$ shown in Fig. 6.7 have the global component vectors

$$\begin{aligned} \mathbf{a} &= \mathbf{r}^A - \mathbf{r} \\ \mathbf{b} &= \mathbf{r}^B - \mathbf{r} \end{aligned} \tag{6.30}$$

and magnitudes a and b, respectively. Vectors $\vec{a}$ and $\vec{b}$ must be orthogonal; i.e., $\mathbf{a}^T\mathbf{b}$ must equal 0. This rule is an important test of the accuracy of the measured data.

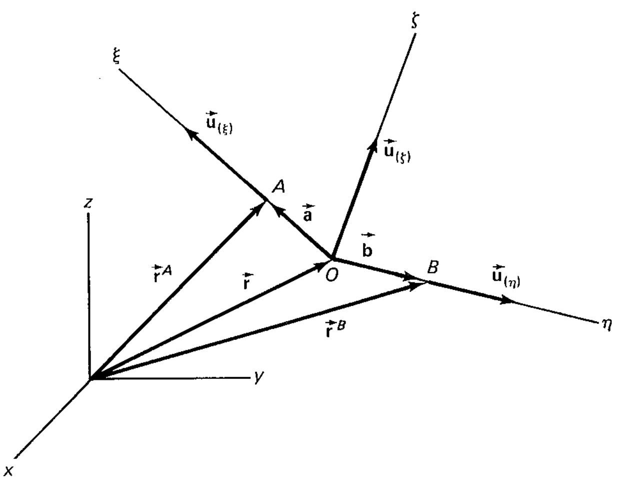

Figure 6.7 Points defining the $\xi\eta\zeta$ axes.

Unit vectors $\vec{u}_{(\xi)}$ and $\vec{u}_{(\eta)}$ may now be defined as follows:

$$\mathbf{u}_{(\xi)} = \frac{\mathbf{a}}{a}$$
$$\mathbf{u}_{(\eta)} = \frac{\mathbf{b}}{b} \tag{6.31}$$

The third unit vector, $\vec{u}_{(\zeta)}$ on the ζ axis, can be found by noting that $\vec{u}_{(\zeta)} = \vec{u}_{(\xi)} \times \vec{u}_{(\eta)}$; i.e.,

$$\mathbf{u}_{(\zeta)} = \tilde{\mathbf{u}}_{(\xi)}\mathbf{u}_{(\eta)} \tag{6.32}$$

Then, from Eq. 6.9, the nine direction cosines are found to be

$$\mathbf{A} = [\mathbf{u}_{(\xi)}, \mathbf{u}_{(\eta)}, \mathbf{u}_{(\zeta)}] \tag{6.33}$$

Example 6.3

Points A on the ξ axis, B on the η axis, and O (the origin of the $\xi\eta\zeta$ axes) have coordinates $\mathbf{r}^A = [0.977, 1.665, 2.916]^T$, $\mathbf{r}^B = [-0.573, 2.539, -0.709]^T$, and $\mathbf{r} = [-0.10, 0.30, 0.25]^T$. Determine the nine direction cosines and the four Euler parameters.

Solution From Eq. 6.30, it is found that

$$\mathbf{a} = [1.077, 1.365, 2.666]^T$$
$$\mathbf{b} = [-0.473, 2.239, -0.959]^T$$

A test for orthogonality shows that $\mathbf{a}^T\mathbf{b} = -0.0099 \simeq 0.0$, which is acceptable. The magnitudes of $\mathbf{a}$ and $\mathbf{b}$ are calculated to be $a = 3.183$ and $b = 2.481$. Then, Eq. 6.31 determines the unit vectors as

$$\mathbf{u}_{(\xi)} = [0.338, 0.429, 0.838]^T$$
$$\mathbf{u}_{(\eta)} = [-0.191, 0.902, -0.387]^T$$

The third unit vector is found from Eq. 6.32:

$$\mathbf{u}_{(\zeta)} = \begin{bmatrix} 0 & -0.838 & 0.429 \\ 0.838 & 0 & -0.338 \\ -0.429 & 0.338 & 0 \end{bmatrix} \begin{bmatrix} -0.191 \\ 0.902 \\ -0.387 \end{bmatrix} = \begin{bmatrix} -0.922 \\ -0.293 \\ 0.387 \end{bmatrix}$$

Hence

$$\mathbf{A} = \begin{bmatrix} 0.338 & -0.191 & -0.922 \\ 0.429 & 0.902 & -0.293 \\ 0.838 & -0.387 & 0.387 \end{bmatrix}$$

which yields, according to the process of Sec. 6.1.4, $\mathbf{p} = [0.810, -0.029, -0.543, 0.191]^T$. The sum of the squares of the four Euler parameters is $\mathbf{p}^T\mathbf{p} = 0.988 \simeq 1$.

In most practical problems, the choice of how to embed a body-fixed coordinate system in a body (a link) is open. The $\xi\eta\zeta$ axes may be embedded in a body according

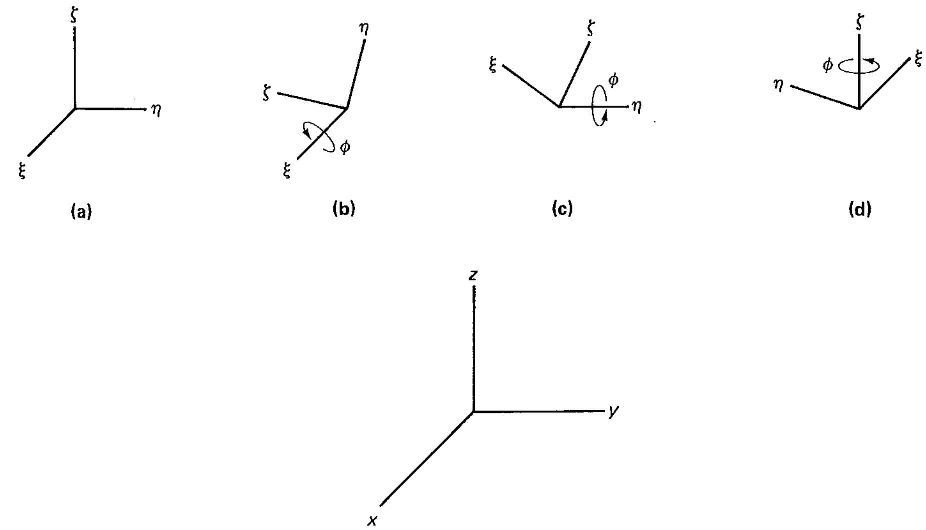

Figure 6.8 Orientation of body-fixed coordinate system in special cases. (a) $\xi\eta\zeta \| xyz$, (b) $\xi \| x$, (c) $\eta \| y$, and (d) $\zeta \| z$.

to any of the configurations shown in Fig. 6.8. If the $\xi\eta\zeta$ axes are parallel to the xyz axes, as shown in Fig. 6.8(a), then

$$\mathbf{p} = [1, 0, 0, 0]^T \qquad \xi\eta\zeta \| xyz \tag{6.34a}$$

If the ξ axis is parallel to the x axis and the angle of rotation is ϕ, as shown in Fig. 6.8(b), then

$$\mathbf{p} = \left[\cos\frac{\phi}{2}, \sin\frac{\phi}{2}, 0, 0\right]^T \qquad \xi \| x \tag{6.34b}$$

Similarly, for the orientations shown in Fig. 6.8(c) and (d),

$$\mathbf{p} = \left[\cos\frac{\phi}{2}, 0, \sin\frac{\phi}{2}, 0\right]^T \qquad \eta \| y \tag{6.34c}$$

and

$$\mathbf{p} = \left[\cos\frac{\phi}{2}, 0, 0, \sin\frac{\phi}{2}\right]^T \qquad \zeta \| z \tag{6.34d}$$

In these special cases, it is relatively simple to determine the angle of rotation and then to calculate the Euler parameters.

6.2 IDENTITIES WITH EULER PARAMETERS

In this section, important formulas and identities between Euler parameters, their time derivatives, and their transformation matrices are derived. Derivation of some of the identities is shown in the text. However, to avoid extensive proofs in the text, several problems are given instead at the end of this chapter. These identities are useful in the derivation of spatial constraint equations and equations of spatial motion.[15]

The product $\mathbf{pp}^T$ is a 4×4 matrix that can be written in the form

$$\mathbf{pp}^T = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} [e_0, \mathbf{e}^T] = \begin{bmatrix} e_0^2 & e_0\mathbf{e}^T \\ e_0\mathbf{e} & \mathbf{ee}^T \end{bmatrix} \tag{6.35}$$

From Eqs. 2.48 and 2.50, it is found that

$$\tilde{\mathbf{e}}\mathbf{e} = \mathbf{0} \tag{6.36}$$

and

$$\tilde{\mathbf{e}}\tilde{\mathbf{e}} = \mathbf{ee}^T - \mathbf{e}^T\mathbf{e}\mathbf{I} = \mathbf{ee}^T - (1 - e_0^2)\mathbf{I} \tag{6.37}$$

A pair of 3×4 matrices $\mathbf{G}$ and $\mathbf{L}$ are defined as†

$$\mathbf{G} = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] \tag{6.38}$$

and

$$\mathbf{L} = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix} = [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}] \tag{6.39}$$

Each row of $\mathbf{G}$ and $\mathbf{L}$ is orthogonal to $\mathbf{p}$; i.e.,

$$\mathbf{Gp} = [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} = [-e_0\mathbf{e} + \tilde{\mathbf{e}}\mathbf{e} + e_0\mathbf{e}] = \mathbf{0} \tag{6.40}$$

where Eq. 6.36 has been used. Similarly,

$$\mathbf{Lp} = \mathbf{0} \tag{6.41}$$

A direct calculation reveals that the rows of $\mathbf{G}$ are orthogonal, as are also the rows of $\mathbf{L}$; i.e.,

$$\mathbf{GG}^T = \mathbf{I} \tag{6.42}$$

and

$$\mathbf{LL}^T = \mathbf{I} \tag{6.43}$$

†These matrices will be used extensively in the formulations that follow in this text. In Sec. 6.4 and some other sections it will be seen that $\mathbf{G}$ and $\mathbf{L}$ are transformation matrices dealing with global and local components of vectors.

so that

$$\mathbf{G}\mathbf{G}^T = \mathbf{L}\mathbf{L}^T \tag{6.44}$$

However, $\mathbf{G}^T\mathbf{G}$ is of the form

$$\begin{aligned}
\mathbf{G}^T\mathbf{G} &= \begin{bmatrix} -\mathbf{e}^T \\ -\tilde{\mathbf{e}} + e_0\mathbf{I} \end{bmatrix} [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] \\
&= \begin{bmatrix} \mathbf{e}^T\mathbf{e} & -\mathbf{e}^T\tilde{\mathbf{e}} - e_0\mathbf{e}^T \\ \tilde{\mathbf{e}}\mathbf{e} - e_0\mathbf{e} & -\tilde{\mathbf{e}}\tilde{\mathbf{e}} + e_0\tilde{\mathbf{e}} - e_0\tilde{\mathbf{e}} + e_0^2\mathbf{I} \end{bmatrix} \\
&= \begin{bmatrix} 1 - e_0^2 & -e_0\mathbf{e}^T \\ -e_0\mathbf{e} & -\mathbf{e}\mathbf{e}^T + \mathbf{I} \end{bmatrix} \\
&= -\begin{bmatrix} e_0^2 & e_0\mathbf{e}^T \\ e_0\mathbf{e} & \mathbf{e}\mathbf{e}^T \end{bmatrix} + \mathbf{I}^* \\
&= -\mathbf{p}\mathbf{p}^T + \mathbf{I}^*
\end{aligned} \tag{6.45}$$

where Eq. 6.37 has been used and $\mathbf{I}^*$ is the 4×4 identity matrix. Similarly, it can be shown that

$$\mathbf{L}^T\mathbf{L} = -\mathbf{p}\mathbf{p}^T + \mathbf{I}^* \tag{6.46}$$

so that

$$\mathbf{G}^T\mathbf{G} = \mathbf{L}^T\mathbf{L} \tag{6.47}$$

A very interesting relationship can be found by evaluating the matrix product $\mathbf{G}\mathbf{L}^T$:

$$\begin{aligned}
\mathbf{G}\mathbf{L}^T &= [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] \begin{bmatrix} -\mathbf{e}^T \\ \mathbf{e} + e_0\mathbf{I} \end{bmatrix} \\
&= \mathbf{e}\mathbf{e}^T + (\tilde{\mathbf{e}} + e_0\mathbf{I})(\tilde{\mathbf{e}} + e_0\mathbf{I}) \\
&= (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})
\end{aligned} \tag{6.48}$$

Comparing Eq. 6.48 with the transformation matrix $\mathbf{A}$ of Eq. 6.18 reveals that

$$\mathbf{A} = \mathbf{G}\mathbf{L}^T \tag{6.49}$$

Equation 6.49 demonstrates that the quadratic transformation matrix $\mathbf{A}$ can be treated as the result of two successive linear transformations. This is one of the most useful relationships between the $\mathbf{G}$ and $\mathbf{L}$ matrices and is a powerful property of Euler parameters.

The first time derivative of Eq. 6.23 yields

$$\mathbf{p}^T\dot{\mathbf{p}} = \dot{\mathbf{p}}^T\mathbf{p} = 0 \tag{6.50}$$

Similarly, the first time derivatives of Eqs. 6.40 and 6.41 result in the identities

$$\mathbf{G}\dot{\mathbf{p}} = -\dot{\mathbf{G}}\mathbf{p} \tag{6.51}$$

and

$$\mathbf{L}\dot{\mathbf{p}} = -\dot{\mathbf{L}}\mathbf{p} \tag{6.52}$$

The product $\dot{\mathbf{G}}\dot{\mathbf{p}}$ may be calculated, using Eq. 6.38, as follows:

$$\begin{aligned}
\dot{\mathbf{G}}\dot{\mathbf{p}} &= [-\dot{\mathbf{e}}, \tilde{\dot{\mathbf{e}}} + \dot{e}_0\mathbf{I}] \begin{bmatrix} \dot{e}_0 \\ \dot{\mathbf{e}} \end{bmatrix} \\
&= -\dot{e}_0\dot{\mathbf{e}} + \dot{e}_0\dot{\mathbf{e}} + \tilde{\dot{\mathbf{e}}}\dot{\mathbf{e}} = \mathbf{0}
\end{aligned} \tag{6.53}$$

since the vector product of $\dot{\mathbf{e}}$ by itself is zero. Similarly,

$$\dot{\mathbf{L}}\dot{\mathbf{p}} = \boldsymbol{0} \tag{6.54}$$

Equation 2.53 can be employed, with Eqs. 6.38 and 6.39, to show that

$$\mathbf{G}\dot{\mathbf{L}}^T = \dot{\mathbf{G}}\mathbf{L}^T \tag{6.55}$$

The time derivative of Eq. 6.49 yields

$$\begin{aligned}\dot{\mathbf{A}} &= \dot{\mathbf{G}}\mathbf{L}^T + \mathbf{G}\dot{\mathbf{L}}^T = 2\mathbf{G}\dot{\mathbf{L}}^T \\ &= 2\dot{\mathbf{G}}\mathbf{L}^T\end{aligned} \tag{6.56}$$

The product $\mathbf{G}\dot{\mathbf{p}}$ can be expanded as follows:

$$\begin{aligned}\mathbf{G}\dot{\mathbf{p}} &= [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix}\dot{e}_0 \\ \dot{\mathbf{e}}\end{bmatrix} \\ &= -\dot{e}_0\mathbf{e} + \tilde{\mathbf{e}}\dot{\mathbf{e}} + e_0\dot{\mathbf{e}}\end{aligned}$$

Transforming both sides of the equation to skew-symmetric matrices, by the operation shown in Eq. 2.43 of Chap. 2, yields

$$\begin{aligned}\widetilde{\mathbf{G}\dot{\mathbf{p}}} &= -\dot{e}_0\tilde{\mathbf{e}} + \widetilde{\tilde{\mathbf{e}}\dot{\mathbf{e}}} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} + \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} - \tilde{\dot{\mathbf{e}}}\tilde{\mathbf{e}} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} + \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} - \mathbf{e}\dot{\mathbf{e}}^T + \dot{\mathbf{e}}^T\mathbf{e}\mathbf{I} + e_0\tilde{\dot{\mathbf{e}}} \\ &= -\dot{e}_0\tilde{\mathbf{e}} + \tilde{\mathbf{e}}\tilde{\dot{\mathbf{e}}} - \mathbf{e}\dot{\mathbf{e}}^T - e_0\dot{e}_0\mathbf{I} + e_0\tilde{\dot{\mathbf{e}}} \\ &= [\mathbf{e}, -\tilde{\mathbf{e}} - e_0\mathbf{I}]\begin{bmatrix}-\dot{\mathbf{e}}^T \\ -\tilde{\dot{\mathbf{e}}} + \dot{e}_0\mathbf{I}\end{bmatrix} \\ &= -\mathbf{G}\dot{\mathbf{G}}^T\end{aligned} \tag{6.57}$$

where Eqs. 2.52, 2.50, and the identity $e_0\dot{e}_0 + \mathbf{e}^T\dot{\mathbf{e}} = 0$ (Eq. 6.50) have been used. Similarly,

$$\widetilde{\mathbf{L}\dot{\mathbf{p}}} = \mathbf{L}\dot{\mathbf{L}}^T \tag{6.58}$$

Two more identities can be derived using Eqs. 6.51, 6.52, 6.57, and 6.58:

$$\mathbf{G}\dot{\mathbf{G}}^T = -\dot{\mathbf{G}}\mathbf{G}^T \tag{6.59}$$

$$\mathbf{L}\dot{\mathbf{L}}^T = -\dot{\mathbf{L}}\mathbf{L}^T \tag{6.60}$$

Furthermore, the time derivative of Eq. 6.50 yields

$$\mathbf{p}^T\ddot{\mathbf{p}} + \dot{\mathbf{p}}^T\dot{\mathbf{p}} = 0 \tag{6.61}$$

The time derivative of Eq. 6.56 results in

$$\begin{aligned}\ddot{\mathbf{A}} &= 2\dot{\mathbf{G}}\dot{\mathbf{L}}^T + 2\mathbf{G}\ddot{\mathbf{L}}^T \\ &= 2\dot{\mathbf{G}}\dot{\mathbf{L}}^T + 2\ddot{\mathbf{G}}\mathbf{L}^T\end{aligned} \tag{6.62}$$

from which it is seen that

$$\ddot{\mathbf{G}}\mathbf{L}^T = \mathbf{G}\ddot{\mathbf{L}}^T \tag{6.63}$$

At this time, it may not be apparent how useful these identities can be. However, later in this chapter and in the next several chapters, these identities will be used extensively.

6.2.1 Identities with Arbitrary Vectors

Additional useful identities between Euler parameters, transformation matrices, and arbitrary vectors are derived here for later use.[15] Consider an arbitrary 3-vector $\mathbf{a}$. Two 4×4 matrices $\overset{+}{\mathbf{a}}$ and $\bar{\mathbf{a}}$ are defined as follows:

$$\overset{+}{\mathbf{a}} \equiv \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \tag{6.64}$$

and

$$\bar{\mathbf{a}} \equiv \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix} \tag{6.65}$$

The overhead plus or minus refers to the sign of the skew-symmetric matrix $\tilde{\mathbf{a}}$ in the definitions. Since $\overset{+}{\mathbf{a}}$ and $\bar{\mathbf{a}}$ are skew-symmetric,

$$\overset{+}{\mathbf{a}}^T = -\overset{+}{\mathbf{a}} \tag{6.66}$$

and

$$\bar{\mathbf{a}}^T = -\bar{\mathbf{a}} \tag{6.67}$$

To illustrate the importance and convenience of this notation, the matrix product $\mathbf{G}^T\mathbf{a}$ may be evaluated as follows:

$$\begin{aligned} \mathbf{G}^T\mathbf{a} &= \begin{bmatrix} -\mathbf{e}^T \\ -\tilde{\mathbf{e}} + e_0\mathbf{I} \end{bmatrix}\mathbf{a} \\ &= \begin{bmatrix} -\mathbf{e}^T\mathbf{a} \\ -\tilde{\mathbf{e}}\mathbf{a} + e_0\mathbf{a} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{a}^T\mathbf{e} \\ \mathbf{a}e_0 + \tilde{\mathbf{a}}\mathbf{e} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix} \\ &= \overset{+}{\mathbf{a}}\mathbf{p} \end{aligned} \tag{6.68}$$

Similarly, it can be shown that

$$\mathbf{L}^T\mathbf{a} = \bar{\mathbf{a}}\mathbf{p} \tag{6.69}$$

The product $\mathbf{G}\overset{+}{\mathbf{a}}$ is evaluated as follows:

$$\begin{aligned} \mathbf{G}\overset{+}{\mathbf{a}} &= [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] \begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & \tilde{\mathbf{a}} \end{bmatrix} \\ &= [\tilde{\mathbf{e}}\mathbf{a} + e_0\mathbf{a}, \mathbf{e}\mathbf{a}^T + \tilde{\mathbf{e}}\tilde{\mathbf{a}} + e_0\tilde{\mathbf{a}}] \\ &= [\tilde{\mathbf{e}}\mathbf{a}, \tilde{\mathbf{a}}\tilde{\mathbf{e}} + e_0\tilde{\mathbf{a}}] + [e_0\mathbf{a}, \mathbf{a}\mathbf{e}^T] \end{aligned} \tag{a}$$

where Eq. 2.53 has been used. It can be shown that $[\tilde{\mathbf{e}}\mathbf{a}, \tilde{\mathbf{a}}\tilde{\mathbf{e}} + e_0\tilde{\mathbf{a}}] = \tilde{\mathbf{a}}\mathbf{G}$, and hence Eq. a is reduced to

$$\mathbf{G}\overset{+}{\mathbf{a}} = \tilde{\mathbf{a}}\mathbf{G} + \mathbf{a}\mathbf{p}^T \tag{6.70}$$

Similarly, it can be shown that

$$\mathbf{L}\bar{\mathbf{a}} = -\tilde{\mathbf{a}}\mathbf{L} + \mathbf{a}\mathbf{p}^T \tag{6.71}$$

The time derivative of Eq. 6.68 can be written as

$$\dot{\mathbf{G}}^T\mathbf{a} + \mathbf{G}^T\dot{\mathbf{a}} = \overset{+}{\dot{\mathbf{a}}}\mathbf{p} + \overset{+}{\mathbf{a}}\mathbf{p}^{\cdot} \qquad (b)$$

Since $\mathbf{a}$ is an arbitrary vector, Eq. 6.68 can be evaluated with the vector $\dot{\mathbf{a}}$, to obtain $\mathbf{G}^T\dot{\mathbf{a}} = \overset{+}{\dot{\mathbf{a}}}\dot{\mathbf{p}}$. This result can be used in Eq. b to obtain

$$\dot{\mathbf{G}}^T\mathbf{a} = \overset{+}{\mathbf{a}}\dot{\mathbf{p}} \qquad (6.72)$$

Similarly, it can be shown that

$$\dot{\mathbf{L}}^T\mathbf{a} = \bar{\mathbf{a}}\dot{\mathbf{p}} \qquad (6.73)$$

Postmultiplying Eq. 6.56 by $\mathbf{a}$ and using Eqs. 6.73 and 6.69 yields

$$\dot{\mathbf{A}}\mathbf{a} = 2\mathbf{G}\,\bar{\mathbf{a}}\,\dot{\mathbf{p}} \qquad (6.74)$$

and

$$\dot{\mathbf{A}}\mathbf{a} = 2\dot{\mathbf{G}}\bar{\mathbf{a}}\mathbf{p} \qquad (6.75)$$

Similarly, it can be shown that

$$\dot{\mathbf{A}}^T\mathbf{a} = 2\mathbf{L}\overset{+}{\mathbf{a}}\dot{\mathbf{p}} \qquad (6.76)$$

and

$$\dot{\mathbf{A}}^T\mathbf{a} = 2\dot{\mathbf{L}}\overset{+}{\mathbf{a}}\mathbf{p} \qquad (6.77)$$

The time derivative of Eq. 6.72 can be written as

$$\ddot{\mathbf{G}}^T\mathbf{a} + \dot{\mathbf{G}}^T\dot{\mathbf{a}} = \overset{+}{\dot{\mathbf{a}}}\dot{\mathbf{p}} + \overset{+}{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.78)$$

Since $\mathbf{a}$ is an arbitrary vector, Eq. 6.72 is also valid as $\dot{\mathbf{G}}^T\dot{\mathbf{a}} = \overset{+}{\dot{\mathbf{a}}}\dot{\mathbf{p}}$. Hence, Eq. 6.78 becomes

$$\ddot{\mathbf{G}}^T\mathbf{a} = \overset{+}{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.79)$$

Similarly, it can be shown that

$$\ddot{\mathbf{L}}^T\mathbf{a} = \bar{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.80)$$

Equation 6.62 is postmultiplied by $\mathbf{a}$ to obtain

$$\ddot{\mathbf{A}}\mathbf{a} = 2\dot{\mathbf{G}}\dot{\mathbf{L}}^T\mathbf{a} + 2\mathbf{G}\ddot{\mathbf{L}}^T\mathbf{a} \qquad (6.81)$$

From Eqs. 6.73 and 6.80, Eq. 6.81 becomes

$$\ddot{\mathbf{A}}\mathbf{a} = 2\dot{\mathbf{G}}\dot{\mathbf{L}}^T\mathbf{a} + 2\mathbf{G}\bar{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.82)$$

or

$$\ddot{\mathbf{A}}\mathbf{a} = 2\dot{\mathbf{G}}\bar{\mathbf{a}}\dot{\mathbf{p}} + 2\mathbf{G}\bar{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.83)$$

Similarly, the product $\ddot{\mathbf{A}}^T\mathbf{a}$ can be written as follows:

$$\ddot{\mathbf{A}}^T\mathbf{a} = 2\dot{\mathbf{L}}\dot{\mathbf{G}}^T\mathbf{a} + 2\mathbf{L}\overset{+}{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.84)$$

or

$$\ddot{\mathbf{A}}^T\mathbf{a} = 2\dot{\mathbf{L}}\overset{+}{\mathbf{a}}\dot{\mathbf{p}} + 2\mathbf{L}\overset{+}{\mathbf{a}}\ddot{\mathbf{p}} \qquad (6.85)$$

The partial derivative of the matrix product $\mathbf{Aa}$ with respect to $\mathbf{p}$ is expanded as follows:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{p}}(\mathbf{Aa}) &= \frac{\partial}{\partial \mathbf{p}}[(2e_0^2 - 1)\mathbf{a} + 2\mathbf{e}\mathbf{e}^T\mathbf{a} + 2e_0\tilde{\mathbf{e}}\mathbf{a}] \\ &= 2[2e_0\mathbf{a} + \tilde{\mathbf{e}}\mathbf{a}, \mathbf{e}^T\mathbf{a}\mathbf{I} + \mathbf{e}\mathbf{a}^T - e_0\tilde{\mathbf{a}}] \qquad (c)\end{aligned}$$

By using Eq. 2.50, we can write this partial derivative thus:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{p}}(\mathbf{Aa}) &= 2[2e_0\mathbf{a} + \tilde{\mathbf{e}}\mathbf{a}, \mathbf{ae}^T - \tilde{\mathbf{e}}\tilde{\mathbf{a}} + \mathbf{ea}^T - e_0\tilde{\mathbf{a}}] \\ &= 2[e_0\mathbf{a} + \tilde{\mathbf{e}}\mathbf{a}, -\tilde{\mathbf{e}}\tilde{\mathbf{a}} + \mathbf{ea}^T - e_0\tilde{\mathbf{a}}] + 2[e_0\mathbf{a}, \mathbf{ae}^T] \\ &= 2[-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}]\begin{bmatrix} 0 & -\mathbf{a}^T \\ \mathbf{a} & -\tilde{\mathbf{a}} \end{bmatrix} + 2\mathbf{ap}^T \\ &= 2\mathbf{G}\bar{\mathbf{a}} + 2\mathbf{ap}^T \end{aligned} \tag{6.86}$$

Similarly, it can be shown that

$$\frac{\partial}{\partial \mathbf{p}}(\mathbf{A}^T\mathbf{a}) = 2\mathbf{L}\overset{+}{\mathbf{a}} + 2\mathbf{ap}^T \tag{6.87}$$

The following identity is valid for the transformation matrix $\mathbf{A}$—which may be described in terms of Euler parameters or any other set of rotational coordinates—and any vector $\vec{s}$. If the vector product of vector $\vec{s}$ and an arbitrary vector $\vec{a}$ is a vector $\vec{b}$, then in terms of global and local components, this vector product is expressed as

$$\mathbf{b} = \tilde{\mathbf{s}}\mathbf{a} \tag{d}$$

and

$$\mathbf{b}' = \tilde{\mathbf{s}}'\mathbf{a}' \tag{e}$$

Since $\mathbf{a} = \mathbf{Aa}'$ and $\mathbf{b} = \mathbf{Ab}'$, Eq. d becomes

$$\mathbf{Ab}' = \tilde{\mathbf{s}}\mathbf{Aa}' \tag{f}$$

Substituting Eq. e into Eq. f and eliminating the arbitrary vector $\mathbf{a}'$ from both sides yields

$$\mathbf{A}\tilde{\mathbf{s}}' = \tilde{\mathbf{s}}\mathbf{A} \tag{6.88}$$

Postmultiplying both sides of Eq. 6.88 by $\mathbf{A}^T$ yields

$$\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}'\mathbf{A}^T \tag{6.89}$$

Equation 6.89 will be found useful in many derivations.

6.3 THE CONCEPT OF ANGULAR VELOCITY

Consider the $\xi\eta\zeta$ coordinate system shown in Fig. 6.9(a), with its origin constrained to the origin of the nonrotating xyz coordinate system, but otherwise free to rotate. The global location of a point P that is fixed in the $\xi\eta\zeta$ coordinate system is given by the equation

$$\mathbf{s}^P = \mathbf{As}'^P$$

Differentiating this equation with respect to time yields

$$\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P + \mathbf{A}\dot{\mathbf{s}}'^P$$

Since $\vec{s}^P$ is fixed in the $\xi\eta\zeta$ axes, $\dot{\mathbf{s}}'^P = \boldsymbol{0}$, and therefore

$$\dot{\mathbf{s}}^P = \dot{\mathbf{A}}\mathbf{s}'^P \tag{6.90}$$

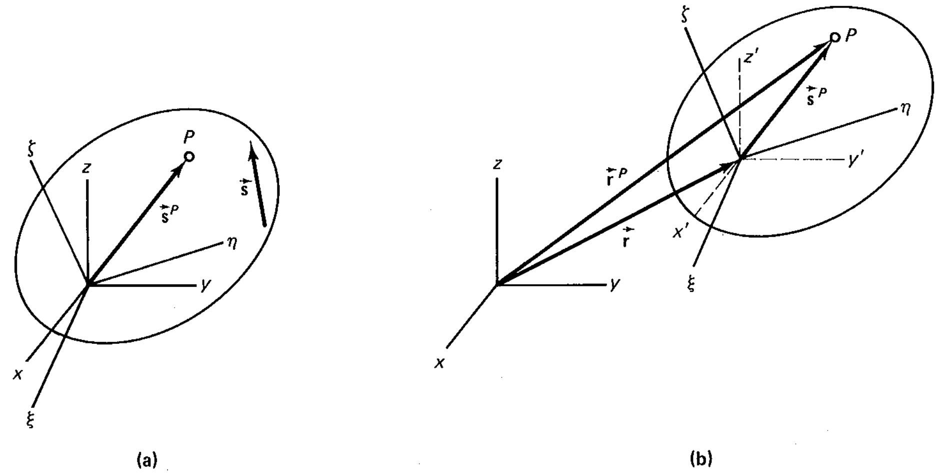

Figure 6.9 (a) Rotating $\xi\eta\zeta$ coordinate system, (b) Rotating and translating $\xi\eta\zeta$ coordinate system.

At this point, the objective is to express the elements of matrix $\dot{\mathbf{A}}$ in terms of the elements of matrix $\mathbf{A}$. Two linear relationships between $\dot{\mathbf{A}}$ and $\mathbf{A}$ may be expressed as

$$\dot{\mathbf{A}} = \mathbf{\Omega A} \tag{6.91}$$

or

$$\dot{\mathbf{A}} = \mathbf{A\Omega}' \tag{6.92}$$

where $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ are two 3×3 coefficient matrices. What the two coefficient matrices are and how they are related will be answered in the remainder of this section.[19]

Differentiating the identity $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ with respect to time yields

$$\dot{\mathbf{A}}^T\mathbf{A} + \mathbf{A}^T\dot{\mathbf{A}} = \mathbf{0} \tag{a}$$

Substituting Eq. 6.91 into Eq. *a* results in

$$\mathbf{A}^T\mathbf{\Omega}^T\mathbf{A} + \mathbf{A}^T\mathbf{\Omega A} = \mathbf{0} \tag{b}$$

Premultiplying Eq. *b* by $\mathbf{A}$ and then postmultiplying the result by $\mathbf{A}^T$ yields $\mathbf{\Omega}^T + \mathbf{\Omega} = \mathbf{0}$, or

$$\mathbf{\Omega} = -\mathbf{\Omega}^T \tag{6.93}$$

Equation 6.93 indicates that $\mathbf{\Omega}$ is a skew-symmetric matrix. Assume that $\mathbf{\Omega}$ is composed of the elements of a 3-vector $\boldsymbol{\omega}$ so that $\mathbf{\Omega} = \tilde{\boldsymbol{\omega}}$. Then Eq. 6.91 becomes

$$\dot{\mathbf{A}} = \tilde{\boldsymbol{\omega}}\mathbf{A} \tag{6.94}$$

Similarly, substituting Eq. 6.92 into Eq. *a* results in

$$\mathbf{\Omega}'^T\mathbf{A}^T\mathbf{A} + \mathbf{A}^T\mathbf{A\Omega}' = \mathbf{0} \tag{c}$$

or

$$\mathbf{\Omega}' = -\mathbf{\Omega}'^T \tag{6.95}$$

Therefore $\mathbf{\Omega}'$ is also a skew-symmetric matrix. Assume that $\mathbf{\Omega}'$ is composed of the elements of a 3-vector $\boldsymbol{\omega}'$ so that $\mathbf{\Omega}' = \tilde{\boldsymbol{\omega}}'$. Then Eq. 6.92 becomes

$$\dot{\mathbf{A}} = \mathbf{A}\tilde{\boldsymbol{\omega}}' \tag{6.96}$$

Comparing Eqs. 6.94 and 6.96 gives

$$\tilde{\boldsymbol{\omega}}\mathbf{A} = \mathbf{A}\tilde{\boldsymbol{\omega}}' \tag{6.97}$$

Equation 6.97 is identical in form with Eq. 6.88; i.e., $\tilde{\mathbf{s}}\mathbf{A} = \mathbf{A}\tilde{\mathbf{s}}'$. Therefore, it can be deduced that $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are the global and the local components of the same vector $\vec{\omega}$. The vector $\vec{\omega}$ is defined as the *angular velocity* of the $\xi\eta\zeta$ coordinate system. The components of vector $\vec{\omega}$ may be expressed as

$$\boldsymbol{\omega} = [\omega_{(x)}, \omega_{(y)}, \omega_{(z)}]^T \tag{6.98}$$

and

$$\boldsymbol{\omega}' = [\omega_{(\xi)}, \omega_{(\eta)}, \omega_{(\zeta)}]^T \tag{6.99}$$

By substituting Eq. 6.94 in Eq. 6.90, it is found that†

$$\begin{aligned} \dot{\mathbf{s}}^P &= \tilde{\boldsymbol{\omega}}\mathbf{A}\mathbf{s}'^P \\ &= \tilde{\boldsymbol{\omega}}\mathbf{s}^P \end{aligned} \tag{6.100}$$

In vector form Eq. 6.100 is expressed as

$$\dot{\vec{s}}^P = \vec{\omega} \times \vec{s}^P$$

For any vector $\vec{s}$ attached to the $\xi\eta\zeta$ coordinate system, like that in Fig. 6.8(a), Eq. 6.100 can be written as

$$\dot{\mathbf{s}} = \tilde{\boldsymbol{\omega}}\mathbf{s} \tag{6.101}$$

For a $\xi\eta\zeta$ coordinate system that rotates and translates relative to the nonmoving xyz axes, the velocity of a point P that is fixed in the $\xi\eta\zeta$ system can be determined. As shown in Fig. 6.9(b), we may employ a translating coordinate system such as $x'y'z'$ whose origin coincides with the origin of the $\xi\eta\zeta$ coordinate axes. The $\xi\eta\zeta$ system rotates relative to the $x'y'z'$ system, which only translates relative to the xyz system. Point P can be located in the xyz system by the relation

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P$$

The time derivative of this equation gives the velocity of point P as

$$\begin{aligned} \dot{\mathbf{r}}^P &= \dot{\mathbf{r}} + \dot{\mathbf{s}}^P \\ &= \dot{\mathbf{r}} + \tilde{\boldsymbol{\omega}}\mathbf{s}^P \end{aligned} \tag{6.102}$$

6.3.1 Time Derivatives of Euler Parameters

In this section, identities between the time derivatives of Euler parameters and angular velocity vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are derived. These identities can be used for conversion from $\boldsymbol{\omega}$ or $\boldsymbol{\omega}'$ to $\dot{\mathbf{p}}$ and vice versa.

Postmultiplying Eq. 6.94 by $\mathbf{A}^T$ yields

$$\dot{\mathbf{A}}\mathbf{A}^T = \tilde{\boldsymbol{\omega}} \tag{6.103}$$

†By substituting Eq. 6.96 in Eq. 6.90, it is found that $\dot{\mathbf{s}}^P = \mathbf{A}\tilde{\boldsymbol{\omega}}'\mathbf{s}'^P$. The global and local components of vector $\dot{\vec{s}}^P$ are denoted by $\dot{\mathbf{s}}^P$ and $(\mathbf{s})'^P$ where $(\dot{\mathbf{s}})'^P = \mathbf{A}(\dot{\mathbf{s}})'^P$, and thus $(\dot{\mathbf{s}})'^P = \tilde{\boldsymbol{\omega}}'\mathbf{s}'^P$. This equation is the same as Eq. 6.100, but expressed in terms of the local components of the vectors. Note that $(\dot{\mathbf{s}})'^P \neq \dot{\mathbf{s}}'^P$. Vector $\dot{\mathbf{s}}'^P$ is defined as the time derivative of a constant vector $\mathbf{s}'^P$, and so $\dot{\mathbf{s}}'^P = \boldsymbol{0}$. However, $(\dot{\mathbf{s}})'^P$ is defined as the local components of vector $\dot{\vec{s}}$, and if $\dot{\vec{s}} \neq \vec{0}$, then $(\dot{\mathbf{s}})'^P$ can be nonzero.

From Eqs. 6.56 and 6.49, Eq. 6.103 becomes $2\dot{\mathbf{G}}\mathbf{L}^T\mathbf{L}\mathbf{G}^T = \tilde{\boldsymbol{\omega}}$, which, upon application of Eqs. 6.46 and 6.40, results in $2\dot{\mathbf{G}}\mathbf{G}^T = \tilde{\boldsymbol{\omega}}$. Finally, substituting Eqs. 6.59 and 6.57 into this last equation gives $2\widetilde{\mathbf{G}\dot{\mathbf{p}}} = \tilde{\boldsymbol{\omega}}$, or

$$\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{p}} \tag{6.104}$$

In expanded form, Eq. 6.104 is

$$\begin{bmatrix} \omega_{(x)} \\ \omega_{(y)} \\ \omega_{(z)} \end{bmatrix} = 2\begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix}\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} \tag{6.105}$$

Premultiplying Eq. 6.104 by $\mathbf{G}^T$ yields $\mathbf{G}^T\boldsymbol{\omega} = 2\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$, which, upon application of Eqs. 6.45 and 6.50, results in the inverse transformation

$$\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\boldsymbol{\omega} \tag{6.106}$$

Similarly, it can be shown that

$$\boldsymbol{\omega}' = 2\mathbf{L}\dot{\mathbf{p}} \tag{6.107}$$

In expanded form, Eq. 6.107 is

$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = 2\begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}\begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} \tag{6.108}$$

The inverse transformation of Eq. 6.107 is found to be

$$\dot{\mathbf{p}} = \tfrac{1}{2}\mathbf{L}^T\boldsymbol{\omega}' \tag{6.109}$$

Differentiating Eq. 6.104 with respect to time yields $\dot{\boldsymbol{\omega}} = 2\mathbf{G}\ddot{\mathbf{p}} + 2\dot{\mathbf{G}}\dot{\mathbf{p}}$, which, upon application of Eq. 6.53, becomes

$$\dot{\boldsymbol{\omega}} = 2\mathbf{G}\ddot{\mathbf{p}} \tag{6.110}$$

Similarly, differentiating Eq. 6.107 with respect to time and using Eq. 6.54 results in

$$\dot{\boldsymbol{\omega}}' = 2\mathbf{L}\ddot{\mathbf{p}} \tag{6.111}$$

Vectors $\dot{\boldsymbol{\omega}}$ and $\dot{\boldsymbol{\omega}}'$ are the global and local components of a vector $\vec{\dot{\omega}}$ defined as the *angular acceleration* of the $\xi\eta\zeta$ coordinate system. It can be shown that the inverses of Eqs. 6.110 and 6.111 are

$$\ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{G}^T\dot{\boldsymbol{\omega}} - \tfrac{1}{4}(\boldsymbol{\omega}^T\boldsymbol{\omega})\mathbf{p} \tag{6.112}$$

and

$$\ddot{\mathbf{p}} = \tfrac{1}{2}\mathbf{L}^T\dot{\boldsymbol{\omega}}' - \tfrac{1}{4}(\boldsymbol{\omega}'^T\boldsymbol{\omega}')\mathbf{p} \tag{6.113}$$

It is clear that $\boldsymbol{\omega}^T\boldsymbol{\omega} = \boldsymbol{\omega}'^T\boldsymbol{\omega}' = \omega^2$, where ω is the magnitude of $\vec{\omega}$. Furthermore, it can be shown that the scalar product $\boldsymbol{\omega}^T\boldsymbol{\omega} - \omega^2 = 0$ yields

$$4\dot{\mathbf{p}}^T\dot{\mathbf{p}} - \omega^2 = 0 \tag{6.114}$$

6.4 SEMIROTATING COORDINATE SYSTEMS

The concept of Euler parameters as rotational coordinates may appear, to the uninitiated reader, as a mathematical tool without any physical meaning. However, careful study of these parameters will prove the contrary. Physical interpretation of Euler parameters is simple and is more natural to implement than any other set of rotational coordinates, such as Euler or Bryant angles.

The angular orientation of one coordinate system relative to another can be looked upon by Euler's theorem as the result of a single rotation about an orientational axis of rotation by an angle ϕ. A viewer may observe a rotation in different ways; three cases are considered here.

Case 1. Consider an observer standing along the axis of rotation in the global xyz system. If the xyz and $\xi\eta\zeta$ coordinates are initially coincident, then as the $\xi\eta\zeta$ system finds its orientation, it will have rotated by an angle ϕ as seen by the observer. A positive rotation may be seen by the observer as a clockwise rotation of $\xi\eta\zeta$ about $\vec{u}$.

Case 2: The observer is in the $\xi\eta\zeta$ coordinate system. In this case the rotation described in case 1 will be viewed as a counterclockwise rotation of the xyz system by an angle ϕ about $\vec{u}$.

Case 3: The observer is in a semirotating coordinate system designated $\alpha\beta\gamma$. In this case the same rotation will be viewed as a clockwise rotation of the $\xi\eta\zeta$ system about $\vec{u}$ by an angle $\phi/2$ and a simultaneous counterclockwise rotation of the xyz system by an angle $\phi/2$.

The three cases are illustrated in Fig. 6.10(a–c) for the special case of a planar system. It is assumed that the axis of rotation is outside the plane, along the z (or ζ or γ) axis. The same example for the general case of a spatial system is illustrated in Fig. 6.10(d–f).

Equation 6.49 states that the transformation matrix **A** is the result of two successive transformations; i.e., **A** can be expressed as the product of two 3×4 matrices **G** and **L** as

$$\mathbf{A} = \mathbf{G}\mathbf{L}^T$$

The components of a vector $\vec{s}$ are transformed from the $\xi\eta\zeta$ coordinate system to the xyz coordinate system as follows:

$$\mathbf{s} = \mathbf{A}\mathbf{s}'$$

This process can be performed in two steps:

$$\mathbf{s}^* = \mathbf{L}^T\mathbf{s}'$$
$$\mathbf{s} = \mathbf{G}\mathbf{S}^*$$

where $\mathbf{s}^*$ is a 4-vector. Matrix $\mathbf{L}^T$ can be interpreted as transforming $\mathbf{s}'$ from the $\xi\eta\zeta$ coordinates by a semirotation to an intermediate 4-space semirotating coordinate system, instead of the 3-space semi-rotating $\alpha\beta\gamma$ system. Hence, $\mathbf{s}^*$ is transformed from the 4-space semirotating system to the xyz system by a second semirotation through matrix **G**.

The transformation matrices **G** and **L** are linear in terms of the Euler parameters. The linearity of **G** and **L** is due to the fact that they perform a coordinate transformation between the local and global systems via a four-dimensional semirotating coordinate system. However, if the semirotating coordinate system is defined in a three-dimen-

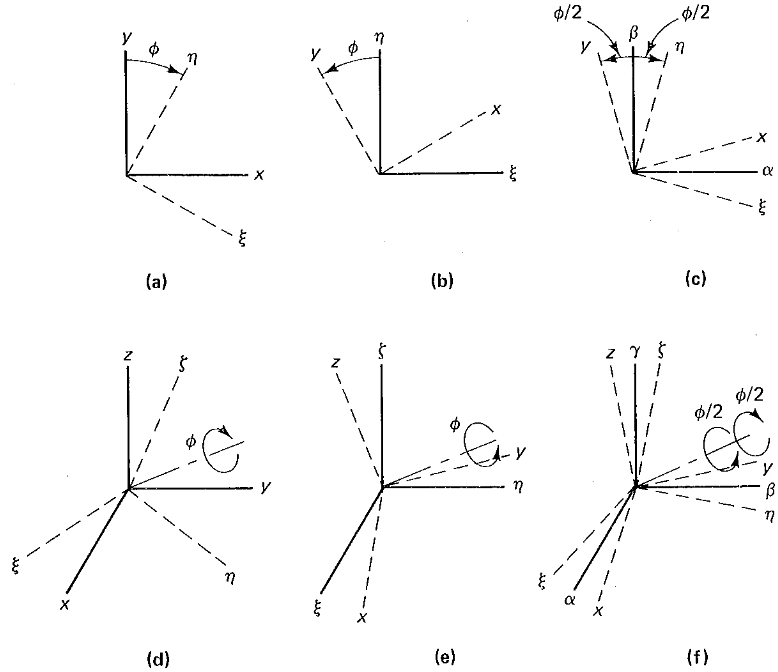

Figure 6.10 Observer's interpretation of angle and orientational axis of rotation as seen from a point on the orientational axis of rotation and in a fixed orientation relative to (a, d) the xyz axes, (b, e) the $\xi\eta\zeta$ axes, and (c, f) the $\alpha\beta\gamma$ axes.

sional space, its corresponding semirotational transformation matrices will be nonlinear in terms of the rotational coordinates.

6.5 RELATIVE AXIS OF ROTATION

The Euler parameters defined in Sec. 6.1.3 describe the angular orientation of a body-fixed coordinate system with respect to a global coordinate system. It may be advantageous to describe the orientation of a body-fixed coordinate system relative to another body-fixed coordinate system. In doing so, we need to find an axis about which one of the coordinate systems may be rotated by some angle to become parallel to the other coordinate system.

Assume that a $\xi_i\eta_i\zeta_i$ coordinate system with respect to the global xyz coordinate system is described by Euler parameters $\mathbf{p}_i$ and transformation matrix $\mathbf{A}_i$. Similarly, assume that the orientation of the $\xi_j\eta_j\zeta_j$ coordinate system with respect to the global coordinate system is described by Euler parameters $\mathbf{p}_j$ and transformation matrix $\mathbf{A}_j$. A vector $\vec{s}$ with $\mathbf{s}_i'$ components in the $\xi_i\eta_i\zeta_i$ system has global components

$$\mathbf{s} = \mathbf{A}_i\mathbf{s}_i' \tag{a}$$

The global components of vector $\vec{s}$, i.e., $\mathbf{s}$, can be transformed in terms of the $\xi_j\eta_j\zeta_j$ coordinate system as follows:

$$\mathbf{s}_j' = \mathbf{A}_j^T\mathbf{s} \tag{b}$$

Substitution of Eq. a into Eq. b yields

$$\mathbf{s}'_j = \mathbf{A}_j^T \mathbf{A}_i \mathbf{s}'_i \tag{c}$$

Equation c may be written as

$$\mathbf{s}'_j = \mathbf{A}_{ij} \mathbf{s}'_i \tag{6.115}$$

where

$$\mathbf{A}_{ij} = \mathbf{A}_j^T \mathbf{A}_i \tag{6.116}$$

The product $\mathbf{A}_j^T \mathbf{A}_i$ or $\mathbf{A}_{ij}$ is the transformation matrix from $\xi_i \eta_i \zeta_i$ coordinates to $\xi_j \eta_j \zeta_j$ coordinates. At this point, the objective is to find a set of Euler parameters

$$\mathbf{p}_{ij} = [e_0, \mathbf{e}^T]_{ij}^T = [e_0, e_1, e_2, e_3]_{ij}^T \tag{6.117}$$

that define the matrix $\mathbf{A}_{ij}$ in terms of Euler parameters $\mathbf{p}_i$ and $\mathbf{p}_j$.

Before attempting to find Euler parameters $\mathbf{p}_{ij}$, we present two identities. The product $\mathbf{L}_i \mathbf{p}_j$ is expanded as follows:

$$\begin{aligned} \mathbf{L}_i \mathbf{p}_j &= [-\mathbf{e}_i, -\tilde{\mathbf{e}}_i + e_{0i}\mathbf{I}] \begin{bmatrix} e_{0j} \\ e_j \end{bmatrix} \\ &= [-e_{0j}\mathbf{e}_i - \tilde{\mathbf{e}}_i \mathbf{e}_j + e_{0i}\mathbf{e}_j] \\ &= [e_{0i}\mathbf{e}_j + \tilde{\mathbf{e}}_j \mathbf{e}_i - e_{0j}\mathbf{e}_i] \\ &= [\mathbf{e}_j, \tilde{\mathbf{e}}_j - e_{0j}\mathbf{I}] \begin{bmatrix} e_{0i} \\ \mathbf{e}_i \end{bmatrix} \\ &= -\mathbf{L}_j \mathbf{p}_i \end{aligned} \tag{6.118}$$

Similarly, it can be shown that

$$\mathbf{G}_i \mathbf{p}_j = -\mathbf{G}_j \mathbf{p}_i \tag{6.119}$$

Now if $\mathbf{A}_i$ in Eq. 6.116 is replaced by $\mathbf{G}_i \mathbf{L}_i^T$ from Eq. 6.49, then postmultiplying by $\mathbf{L}_i$ yields

$$\begin{aligned} \mathbf{A}_{ij}\mathbf{L}_i &= \mathbf{A}_j^T \mathbf{G}_i \mathbf{L}_i^T \mathbf{L}_i \\ &= \mathbf{A}_j^T \mathbf{G}_i (-\mathbf{p}_i \mathbf{p}_i^T + \mathbf{I}^*) \\ &= \mathbf{A}_j^T \mathbf{G}_i \end{aligned} \tag{d}$$

where Eqs. 6.45 and 6.40 have been used. Postmultiplying Eq. d by $\mathbf{p}_j$ yields

$$\begin{aligned} \mathbf{A}_{ij}\mathbf{L}_i \mathbf{p}_j &= \mathbf{A}_j^T \mathbf{G}_i \mathbf{p}_j \\ &= -\mathbf{L}_j \mathbf{G}_j^T \mathbf{G}_j \mathbf{p}_i \\ &= -\mathbf{L}_j (-\mathbf{p}_j \mathbf{p}_j^T + \mathbf{I}^*)\mathbf{p}_i \\ &= -\mathbf{L}_j \mathbf{p}_i \end{aligned}$$

or

$$\mathbf{A}_{ij}\mathbf{L}_j \mathbf{p}_i = \mathbf{L}_j \mathbf{p}_i \tag{e}$$

where Eqs. 6.49, 6.119, 6.45, 6.41, and 6.118 have been employed, in that order. Equation e may be rewritten as

$$\mathbf{A}_{ij}\mathbf{b} = \mathbf{b} \tag{f}$$

where

$$\mathbf{b} = \mathbf{L}_j \mathbf{p}_i \tag{g}$$

Equation g shows that the transformation matrix $\mathbf{A}_{ij}$ does not change the components of vector $\vec{b}$. Therefore, vector $\vec{b}$ must be located along a *relative orientational axis of rotation* between the $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ coordinate systems. Since vector $\vec{e}_{ij}$ also lies along the same relative orientational axis of rotation, $\vec{e}_{ij}$ and $\vec{b}$ must be collinear.

According to Eq. 6.25, the Euler parameter e_{0ij} can be evaluated from

$$e_{0ij}^2 = \frac{\text{tr } \mathbf{A}_{ij} + 1}{4} \tag{h}$$

The trace of matrix $\mathbf{A}_{ij}$ can be found by substituting the elements of matrices $\mathbf{A}_i$ and $\mathbf{A}_j$ from Eq. 6.19 into Eq. 6.116. If the matrix product is carried out and the trace of the resultant 3×3 matrix is formed and simplified,† the trace of $\mathbf{A}_{ij}$ is found to be

$$\text{tr } \mathbf{A}_{ij} = 4(\mathbf{p}_j^T\mathbf{p}_i)^2 - 1 \tag{i}$$

Substitution of Eq. i into Eq. h yields

$$e_{0ij}^2 = (\mathbf{p}_j^T\mathbf{p}_i)^2$$

or

$$e_{0ij} = \mathbf{p}_j^T\mathbf{p}_i \tag{6.120}$$

where, according to the discussion of Sec. 6.1.4, the positive sign is chosen.

Calculating the sum of the squares of e_{0ij} and the components of vector $\mathbf{b}$ reveals that

$$\begin{aligned} e_{0ij}^2 + \mathbf{b}^T\mathbf{b} &= (\mathbf{p}_j^T\mathbf{p}_i)^2 + \mathbf{p}_i^T\mathbf{L}_j^T\mathbf{L}_j\mathbf{p}_i \\ &= (\mathbf{p}_j^T\mathbf{p}_i)^2 + \mathbf{p}_i^T(-\mathbf{p}_j\mathbf{p}_j^T + \mathbf{I}^*)\mathbf{p}_i \\ &= (\mathbf{p}_j^T\mathbf{p}_i)^2 - (\mathbf{p}_j^T\mathbf{p}_i)^2 + \mathbf{p}_i^T\mathbf{p}_i \\ &= 1 \end{aligned} \tag{j}$$

where Eqs. 6.46, 2.41, and 6.23 have been used. Since it is already known that $\mathbf{e}_{ij}$ and $\mathbf{b}$ are parallel, then a comparison of Eq. j and Eq. 6.21 indicates that $\mathbf{e}_{ij} = \mathbf{b}$, or

$$\mathbf{e}_{ij} = \mathbf{L}_j\mathbf{p}_i \tag{6.121}$$

Hence, the Euler parameters $\mathbf{p}_{ij}$ are

$$\mathbf{p}_{ij} = \begin{bmatrix} e_0 \\ \mathbf{e} \end{bmatrix}_{ij} = \begin{bmatrix} \mathbf{p}_j^T\mathbf{p}_i \\ \mathbf{L}_j\mathbf{p}_i \end{bmatrix} = \begin{bmatrix} \mathbf{p}^T \\ \mathbf{L} \end{bmatrix}_j \mathbf{p}_i$$

or

$$\mathbf{p}_{ij} = \mathbf{L}_j^*\mathbf{p}_i \tag{6.122}$$

where $\mathbf{L}_j^*$ is a 4×4 matrix defined as

$$\mathbf{L}_j^* = \begin{bmatrix} \mathbf{p}^T \\ \mathbf{L} \end{bmatrix}_j \tag{6.123}$$

Equation 6.122 shows that if the Euler parameters describing the orientations of two bodies with respect to a global coordinate system are known, then the Euler parameters describing the orientation of one of the bodies with respect to the other can be determined.

Two more identities are stated here that can be verified easily:

$$\mathbf{L}_i\dot{\mathbf{p}}_j = -\dot{\mathbf{L}}_j\mathbf{p}_i \tag{6.124}$$

†Since the calculation of tr $\mathbf{A}_{ij}$ is too extensive to be listed in detail, only the final result is presented.

and

$$\mathbf{G}_i\dot{\mathbf{p}}_j = -\dot{\mathbf{G}}_j\mathbf{p}_i \tag{6.125}$$

These identities relate the Euler parameters of one body and the time derivative of the Euler parameters of another body.

6.5.1 Intermediate Axis of Rotation

In Section 6.5, a relative orientational axis of rotation between $\xi_i\eta_i\zeta_i$ and $\xi_j\eta_j\zeta_j$ coordinate systems was found when the Euler parameters that describe the orientation of the two systems with respect to the global coordinate system were known. This method can be stated in another form, but identical in principle, as follows:

> *Find an* intermediate orientational axis of rotation *about which a body-fixed coordinate system at time* t^k *can be rotated to become parallel to the coordinate system describing the orientation of the same body at time* t^l.

If the coordinate system of body i at times t^k and t^l is denoted by $\xi_i\eta_i\zeta_i^k$ and $\xi_i\eta_i\zeta_i^l$, the Euler parameters of the body at t^k and t^l are denoted by $\mathbf{p}_i^k$ and $\mathbf{p}_i^l$, respectively. Similarly, the intermediate set of Euler parameters between orientations at t^k and t^l, which is denoted by $\mathbf{p}_i^{kl}$, can be written from Eq. 6.122 as

$$\mathbf{p}_i^{kl} = \mathbf{L}_i^{*l}\mathbf{p}_i^k \tag{6.126}$$

where

$$\mathbf{L}_i^{*l} = \begin{bmatrix} \mathbf{p}^T \\ \mathbf{L} \end{bmatrix}_i^l \tag{6.127}$$

6.6 FINITE ROTATION

Consider the two bodies i and j shown in Fig. 6.11 (these may also be interpreted as two different configurations of the same body). The translational vectors for the two bodies are $\mathbf{r}_i$ and $\mathbf{r}_j$, and the translational vector between the two bodies is denoted by $\mathbf{r}_{ij}$. It is clear that

$$\begin{aligned} \mathbf{r}_j &= \mathbf{r}_i + \mathbf{r}_{ji} \\ &= \mathbf{r}_{ji} + \mathbf{r}_i \end{aligned}$$

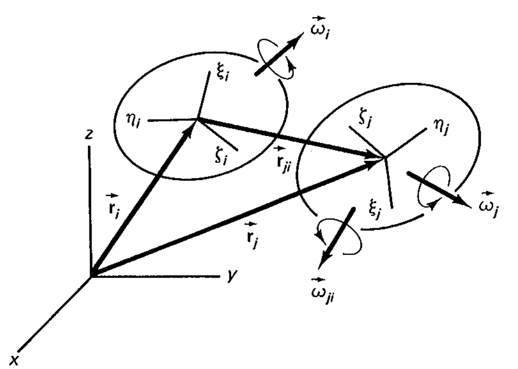

Figure 6.11 Two bodies with different translational and rotational configurations.

This indicates that translational vectors follow the commutative law of vector summation, and therefore a translational vector is a true vector quantity. However, it will be seen that this is not a characteristic of the rotation of a body. If the transformation matrices for the two bodies are $\mathbf{A}_i$ and $\mathbf{A}_j$ and the transformation matrix of body j with respect to body i is $\mathbf{A}_{ji}$, then from Eq. 6.116 (by reversing the indices) it is found that $\mathbf{A}_{ji} = \mathbf{A}_i^T\mathbf{A}_j$, or

$$\begin{aligned}\mathbf{A}_j &= \mathbf{A}_i\mathbf{A}_{ji}\\ &\neq \mathbf{A}_{ji}\mathbf{A}_i\end{aligned} \tag{a}$$

This is obvious, since matrix multiplication is not commutative. This means that in two successive rotations, the order of rotations cannot be reversed. Assume that a finite rotation is denoted by a rotational vector in the direction of the orientational axis of rotation, having a magnitude proportional to the angle of rotation, e.g., $\mathbf{e} = \mathbf{u} \sin \phi/2$. Then three rotational vectors $\mathbf{e}_i$, $\mathbf{e}_j$, and $\mathbf{e}_{ji}$ can be defined and it can be deduced that

$$\mathbf{e}_j \neq \mathbf{e}_i + \mathbf{e}_{ji}$$

In contrast to the rotational vector of a finite rotation, the angular velocity vector is a true vector quantity. The time derivative of Eq. a is

$$\dot{\mathbf{A}}_j = \dot{\mathbf{A}}_i\mathbf{A}_{ji} + \mathbf{A}_i\dot{\mathbf{A}}_{ji} \tag{b}$$

From Eq. 6.94, Eq. b becomes

$$\tilde{\boldsymbol{\omega}}_j\mathbf{A}_j = \tilde{\omega}_i\mathbf{A}_i\mathbf{A}_{ji} + \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_{ji}\mathbf{A}_{ji} \tag{c}$$

Note that $\tilde{\boldsymbol{\omega}}'_{ji}$ represents the components of $\tilde{\omega}_{ji}$ with respect to the $\xi_j\eta_j\zeta_j$ coordinate system. Substituting Eq. a in Eq. c and simplifying the result yields

$$\begin{aligned}\tilde{\boldsymbol{\omega}}_j &= \tilde{\boldsymbol{\omega}}_i + \mathbf{A}_i\tilde{\boldsymbol{\omega}}'_{ji}\mathbf{A}_i^T\\ &= \tilde{\boldsymbol{\omega}}_i + \tilde{\boldsymbol{\omega}}_{ji}\end{aligned}$$

Therefore,

$$\boldsymbol{\omega}_j = \boldsymbol{\omega}_i + \boldsymbol{\omega}_{ji}$$

This is the proof that the angular velocity is a true vector quantity.

PROBLEMS

6.1 Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are defined along the positive ξ axis, η axis, and ζ axis, respectively. The global components of these vectors are

$$\mathbf{a} = \begin{bmatrix} 0.0776 \\ -1.8833 \\ -0.6685 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.6410 \\ 1.0038 \\ -2.7535 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 1.4642 \\ -0.0537 \\ 0.3213 \end{bmatrix}$$

(a) Test these three vectors for orthogonality.

(b) Determine the global components of the three unit vectors $\mathbf{u}_{(\xi)}$, $\mathbf{u}_{(\eta)}$, and $\mathbf{u}_{(\zeta)}$ along the $\xi\eta\zeta$ axes.

(c) Determine the nine direction cosines of matrix $\mathbf{A}$.

6.2 Using Eq. 6.3, find six constraint equations between the nine direction cosines.

6.3 Two vectors $\vec{a}$ and $\vec{b}$ are defined along the positive ξ and η axes, respectively. The global components of these vectors are

$$\mathbf{a} = \begin{bmatrix} 0.1107 \\ 0.3924 \\ 1.1286 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -1.9450 \\ 1.5330 \\ -0.3422 \end{bmatrix}$$

(a) Test these two vectors for orthogonality.

(b) Determine the global components of the three unit vectors $\mathbf{u}_{(\xi)}$, $\mathbf{u}_{(\eta)}$, and $\mathbf{u}_{(\zeta)}$ along the $\xi\eta\zeta$ axes.

(c) Determine the elements of matrix **A**.

6.4 Two vectors $\vec{a}$ and $\vec{c}$ are defined along the positive ξ and ζ axes, respectively. The global components of these vectors are

$$\mathbf{a} = \begin{bmatrix} 0.6438 \\ 2.3930 \\ -1.6909 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} -0.7796 \\ -0.2077 \\ -0.5908 \end{bmatrix}$$

Determine the elements of matrix **A**.

6.5 A vector $\vec{a}$ along the positive ξ axis and a vector $\vec{d}$ on the $\xi\eta$ plane have the following components:

$$\mathbf{a} = \begin{bmatrix} -1.0 \\ 1.2 \\ 0.5 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} 1.3 \\ -0.6 \\ 0.8 \end{bmatrix}$$

(a) Determine the three unit vectors along the $\xi\eta\zeta$ axes.

(b) Find the elements of matrix **A**.

(c) Is the solution to this problem unique?

6.6 Determine the four Euler parameters for the transformation matrices **A** in

(a) Prob. 6.1

(b) Prob. 6.3

(c) Prob. 6.4

(d) Prob. 6.5

6.7 Determine the four Euler parameters for the transformation matrices

(a)

$$\mathbf{A} = \begin{bmatrix} -0.4590 & 0.8376 & -0.2962 \\ 0.4908 & 0.5170 & 0.7014 \\ 0.7406 & 0.1766 & -0.6483 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} -0.4590 & 0.4908 & 0.7406 \\ 0.8376 & 0.5170 & 0.1766 \\ -0.2962 & 0.7014 & -0.6483 \end{bmatrix}$$

Compare the results of parts (a) and (b). What do you conclude?

6.8 A transformation matrix **A** is given as follows:

$$\mathbf{A} = \begin{bmatrix} 0.0319 & -0.8506 & 0.5249 \\ -0.8506 & -0.2988 & -0.4327 \\ 0.5249 & -0.4327 & -0.7330 \end{bmatrix}$$

(a) Test matrix **A** for orthogonality, using the identity $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

(b) Determine the four Euler parameters for matrix **A**.

6.9 By using Eq. 6.18, show that if the signs of all four Euler parameters are reversed, i.e., if $\mathbf{p} \to -\mathbf{p}$, then the transformation matrix **A** is not affected.

6.10 If the angular orientation of a body-fixed coordinate system is described in terms of three Euler angles $\psi = 40°$, $\theta = 30°$, and $\sigma = -25°$, find its corresponding set of Euler parameters.

6.11 Determine the global coordinates of the point $[2, 6, 8]^T$ in a rotating $\xi\eta\zeta$ system, where the Euler angles are $\psi = 45°$, $\theta = 45°$, and $\sigma = 30°$ and the origins of the two coordinate systems coincide.

6.12 Determine the coordinates of a point in a rotating $\xi\eta\zeta$ system if its global coordinates are $[3, 3, 1]^T$. The Euler angles are $\psi = 30°$, $\theta = 30°$, and $\sigma = 60°$ and the origins of the two systems coincide.

6.13 A sequence of two rotations is required to uniquely locate the longitudinal axis of a vehicle. Consider the sequence ψ, θ as shown in Fig. P.6.13. The first rotation is a positive rotation about the x axis through an angle ψ; the second, a positive rotation about the ζ' axis through an angle θ.

(a) Determine the elements of a transformation matrix **A**.

(b) Test **A** for orthogonality.

6.14 Find the Euler angles describing the rotation shown in Fig. P.6.14 for $\varepsilon = 30°$.

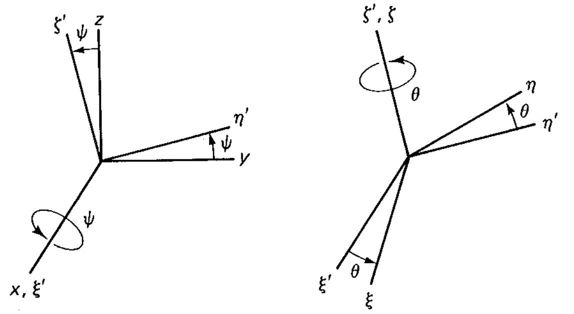

Figure P.6.13

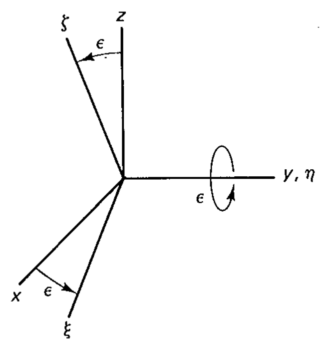

Figure P.6.14

6.15 Consider the spinning top shown in Fig. P.6.15. Assume Euler angle rates of $\dot{\psi} = 2$ rad/s, $\dot{\theta} = 0$, and $\dot{\sigma} = 125$ rad/s when $\psi = 120°$, $\theta = 30°$, and $\sigma = 90°$.

(a) Determine the corresponding values of $\omega_{(\xi)}$, $\omega_{(\eta)}$, and $\omega_{(\zeta)}$.

(b) For what values of θ would the inverse of the transformation in (a) be nonexistent?

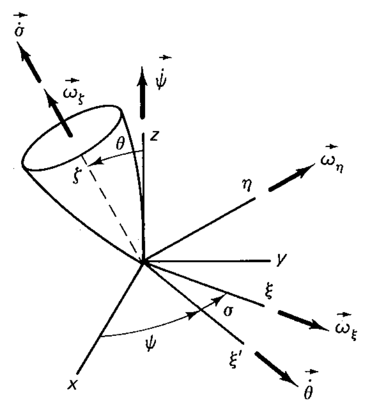

Figure P.6.15

6.16 For the spinning top of Prob. 6.15, assume angular rate components of $\omega_{(\xi)} = 0$, $\omega_{(\eta)} = 2$ rad/s, and $\omega_{(\zeta)} = 20$ rad/s when $\psi = 60°$, $\theta = 30°$, and $\sigma = 120°$.

(a) Determine $\omega_{(x)}$, $\omega_{(y)}$, and $\omega_{(z)}$.

(b) Determine $\dot\psi$, $\dot\theta$, and $\dot\sigma$.

6.17 Find expressions to convert Bryant angles into equivalent Euler parameters.

6.18 Find expressions to convert Euler parameters to Bryant angles.

6.19 Show that the rotation angle ϕ and the components of the unit vector **u** along the orientational axis of rotation are determined by the equations

$$\cos\phi = \frac{\text{tr}\,\mathbf{A} - 1}{2}$$

$$u_i^2 = \frac{a_{ii} - \cos\phi}{1 - \cos\phi} \qquad i = 1, 2, 3$$

where a_{ii} is an element of **A**.

6.20 Point P is located from the origin of a coordinate system by a vector having components $[3, 4, 12]^T$. The vector is subjected to a rotation of 30° positive about an axis passing through the origin and a point $[2, -3, 2]^T$. Find the components of the vector in its new orientation.

6.21 The Euler parameters describing the orientation of a body-fixed coordinate system with respect to a global coordinate system are $e_0 = e_3 = 0.6533$, $e_1 = 0.3827$, and $e_2 = 0$. Find the three Euler angles describing this orientation.

6.22 Four Euler parameters $\mathbf{p} = [0.8, 0, 0, 0.6]^T$ describe the angular orientation of the $\xi\eta\zeta$ axes with respect to the xyz axes. Determine the equivalent set of Euler angles for this orientation. What do you conclude?

6.23 Points D and F are located on the η and ζ axes, respectively, and have local coordinates $\mathbf{s}'^D = [0, 2, 0]^T$ and $\mathbf{s}'^F = [0, 0, 1]^T$. The global components of vectors $\vec{s}^D$ and $\vec{s}^F$ are $\mathbf{s}^D = [-1.1098, 1.6598, -0.1134]^T$ and $\mathbf{s}^F = [0.6285, 0.4629, 0.6249]^T$. Find the elements of matrix **A** describing this orientation.

6.24 Verify Eq. 6.41.

6.25 Verify Eq. 6.42 (or 6.43),

(a) Using the expanded form of matrix **G** (or **L**)

(b) Using the compact form of matrix **G** (or **L**)

6.26 Verify the following identities:

- **(a)** Eq. 6.46
- **(b)** Eq. 6.54
- **(c)** Eq. 6.55
- **(d)** Eq. 6.58
- **(e)** Eq. 6.59
- **(f)** Eq. 6.60

6.27 Verify the following identities:

- **(a)** Eq. 6.69
- **(b)** Eq. 6.71
- **(c)** Eq. 6.73
- **(d)** Eq. 6.80
- **(e)** Eqs. 6.84 and 6.85
- **(f)** Eq. 6.87

6.28 Start with Eq. 6.96 and obtain Eq. 6.107.

6.29 Show that the inverse transformation of Eq. 6.109 is valid.

6.30 Verify the following identities:

- **(a)** Eq. 6.112
- **(b)** Eq. 6.113
- **(c)** Eq. 6.119

6.31 Show that tr $\mathbf{A}_{ij} = 4(\mathbf{p}_j^T\mathbf{p}_i)^2 - 1$ by determining the diagonal elements of the matrix product $\mathbf{A}_j^T\mathbf{A}_i$.

6.32 Derive the inverse transformation of Eq. 6.122; i.e., calculate $\mathbf{p}_i$ when $\mathbf{p}_j$ and $\mathbf{p}_{ij}$ are known (*Hint:* Start with Eq. 6.121).