2

Vectors and Matrices

Vector and matrix algebra form the mathematical foundation for kinematics and dynamics. Geometry of motion is at the heart of both the kinematics and the dynamics of mechanical systems. Vector analysis is the time-honored tool for describing geometry. In its *geometric form,* however, vector algebra is not well suited to computer implementation.

In this chapter, a systematic matrix formulation of vector algebra, referred to as *algebraic vector representation,* is presented for use throughout the text. This form of vector representation, in contrast to the more traditional geometric form of vector representation, is easier to use for either formula manipulation or computer implementation. Elementary properties of vector and matrix algebra are stated in this chapter without proof.

2.1 GEOMETRIC VECTOR

When we write a vector in the form $\vec{a}$, it is understood from the arrow notation that we are referring to the vector in its geometric sense: it begins at a point A and ends at a point B. The *magnitude* of vector $\vec{a}$ is denoted by a. A *unit vector* in the direction of $\vec{a}$ is shown as $\vec{u}_{(a)}$.

Vectors lying in the same plane are called *coplanar vectors*. *Collinear vectors* have the same direction and the same line of action. *Equal vectors* have the same magnitude and direction. A *zero* or *null vector* has zero magnitude and therefore no specified direction.

Multiplication of a vector $\vec{a}$ by a scalar α is defined as a vector in the same direction as $\vec{a}$ that has a magnitude αa. The negative of a vector is obtained by multiplying the vector by -1; it changes the direction of the vector.

The *vector sum* of two vectors $\vec{a}$ and $\vec{b}$ is written as

$$\vec{c} = \vec{a} + \vec{b} \tag{2.1}$$

The product of a sum of two scalars $\alpha + \beta$ and a vector $\vec{a}$ is expanded as

$$(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a} \tag{2.2}$$

A vector $\vec{a}$ can be resolved into its *Cartesian components* $a_{(x)}$, $a_{(y)}$, and $a_{(z)}$ along the x, y, and z axes of a Cartesian system. Here, the unit vectors $\vec{u}_{(x)}$, $\vec{u}_{(y)}$, and $\vec{u}_{(z)}$ are directed along the coordinate axes x, y, and z.[†] In vector notation, the resolution of the vector into its components is expressed as

$$\vec{a} = a_{(x)}\vec{u}_{(x)} + a_{(y)}\vec{u}_{(y)} + a_{(z)}\vec{u}_{(z)} \tag{2.3}$$

If the angles between the vector $\vec{a}$ and the positive x, y, and z axes are denoted by $\theta_{(x)}$, $\theta_{(y)}$, and $\theta_{(z)}$, the components of vector $\vec{a}$ are given as

$$\begin{aligned} a_{(x)} &= a \cos \theta_{(x)} \\ a_{(y)} &= a \cos \theta_{(y)} \\ a_{(z)} &= a \cos \theta_{(z)} \end{aligned} \tag{2.4}$$

The quantities $\cos \theta_{(x)}$, $\cos \theta_{(y)}$, and $\cos \theta_{(z)}$ are the *direction cosines* of vector $\vec{a}$.

The *scalar* (or *dot*) *product* of two vectors $\vec{a}$ and $\vec{b}$ is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them; i.e.,

$$\vec{a} \cdot \vec{b} = ab \cos \theta \tag{2.5}$$

$$= a_{(x)}b_{(x)} + a_{(y)}b_{(y)} + a_{(z)}b_{(z)} \tag{2.6}$$

$$= \vec{b} \cdot \vec{a} \tag{2.7}$$

where the angle θ between the vectors is measured in the plane of intersection of the vectors. If the two vectors are nonzero, i.e., if $a \neq 0$ and $b \neq 0$, then their scalar product is zero only if $\cos \theta = 0$. Two nonzero vectors are thus said to be *orthogonal* (perpendicular) if their scalar product is zero. For any vector $\vec{a}$,

$$\vec{a} \cdot \vec{a} = a^2 \tag{2.8}$$

The *vector* (or *cross*) *product* of two vectors $\vec{a}$ and $\vec{b}$ is defined as the vector

$$\vec{c} = \vec{a} \times \vec{b} \tag{2.9}$$

$$= ab \sin \theta\, \vec{u} \tag{2.10}$$

$$= (a_{(y)}b_{(z)} - a_{(z)}b_{(y)})\vec{u}_{(x)} + (a_{(z)}b_{(x)} - a_{(x)}b_{(z)})\vec{u}_{(y)} + (a_{(x)}b_{(y)} - a_{(y)}b_{(x)})\vec{u}_{(z)} \tag{2.11}$$

where $\vec{u}$ is a unit vector that is orthogonal to the plane of intersection of the two vectors $\vec{a}$ and $\vec{b}$, taken in the positive right-hand coordinate direction, and θ is the angle between vectors a and b. Since reversal of the order of the vectors $\vec{a}$ and $\vec{b}$ in Eq. 2.9 would yield an opposite direction for the unit vector, it is clear that

$$\vec{b} \times \vec{a} = -\vec{a} \times \vec{b} \tag{2.12}$$

While not obvious on geometrical grounds, the following identities are valid:

$$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} \tag{2.13}$$

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \tag{2.14}$$

[†]In most textbooks the three unit vectors along the x, y, and z axes are denoted by $\vec{i}$, $\vec{j}$, and $\vec{k}$. In this text, since i and j are used to denote indices of bodies, to avoid any confusion, unit vectors are denoted by $\vec{u}_{(x)}$, $\vec{u}_{(y)}$, and $\vec{u}_{(z)}$.

From the definition of scalar product, vector product, and unit coordinate vectors, the following identities are valid:

$$\begin{aligned} \vec{u}_{(x)} \cdot \vec{u}_{(y)} &= \vec{u}_{(y)} \cdot \vec{u}_{(z)} = \vec{u}_{(z)} \cdot \vec{u}_{(x)} = 0 \\ \vec{u}_{(x)} \cdot \vec{u}_{(x)} &= \vec{u}_{(y)} \cdot \vec{u}_{(y)} = \vec{u}_{(z)} \cdot \vec{u}_{(z)} = 1 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \vec{u}_{(x)} \times \vec{u}_{(x)} &= \vec{u}_{(y)} \times \vec{u}_{(y)} = \vec{u}_{(z)} \times \vec{u}_{(z)} = \vec{0} \\ \vec{u}_{(x)} \times \vec{u}_{(y)} &= \vec{u}_{(z)} \\ \vec{u}_{(y)} \times \vec{u}_{(z)} &= \vec{u}_{(x)} \\ \vec{u}_{(z)} \times \vec{u}_{(x)} &= \vec{u}_{(y)} \end{aligned} \tag{2.16}$$

2.2 MATRIX AND ALGEBRAIC VECTORS

Compact matrix notation often allows one to concentrate on the form of a system of equations and what it means, rather than on the minute details of its construction. Matrix manipulation also allows for the organized development and simplification of systems of equations.

A matrix with m rows and n columns is said to be of *dimension* $m \times n$ and is denoted by a boldface capital letter; it is written in the form

$$\mathbf{A} \equiv [a_{ij}] \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{(m \times n)}$$

where a typical element a_{ij} is located at the intersection of the ith row and jth column. The *transpose of a matrix* is formed by interchanging rows and columns and is designated by the superscript T. Thus, if a_{ij} is the ij element of matrix $\mathbf{A}$, a_{ji} is the ij element of its transpose $\mathbf{A}^T$.

A matrix with only one column is called a *column matrix* and is denoted by a boldface lower-case letter; e.g., $\mathbf{a}$. A matrix with only one row is called a *row matrix* and is denoted as $\mathbf{a}^T$; i.e., as the transpose of a column matrix. An $m \times n$ matrix can be considered to be constructed of n column matrices $\mathbf{a}_j$, where $j = 1, \ldots, n$, or m row matrices $\mathbf{a}_i^T$; where $i = 1, \ldots, m$.

The vector $\vec{a}$ in Eq. 2.3 is uniquely defined by its Cartesian components and can be written in matrix notation as follows:

$$\mathbf{a} = \begin{bmatrix} a_{(x)} \\ a_{(y)} \\ a_{(z)} \end{bmatrix} \equiv [a_{(x)}, a_{(y)}, a_{(z)}]^T \tag{2.17}$$

This is the *algebraic* (or *component*) *representation of a vector.*

2.2.1 Matrix Operations

In this section, the terminology of matrix algebra is briefly reviewed. Several useful identities are stated that are used extensively throughout this text.

A *square matrix* has an equal number of rows and columns. A *diagonal matrix* is a square matrix with $a_{ij} = 0$ for $i \neq j$ and at least one nonzero diagonal term. An $n \times n$ diagonal matrix is denoted by

$$\mathbf{A} \equiv \text{diag}[a_{11}, a_{22}, \ldots, a_{nn}]$$

If square matrices $\mathbf{B}_i$, $i = 1, \ldots, k$, are arranged along the diagonal of a matrix $\mathbf{D}$ to give

$$\mathbf{D} \equiv \begin{bmatrix} \mathbf{B}_1 & & & \\ & \mathbf{B}_2 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \mathbf{B}_k \end{bmatrix}$$

then the matrix is called a *quasi-diagonal matrix* and is denoted by

$$\mathbf{D} \equiv \text{diag}[\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_k]$$

even though $\mathbf{D}$ is not a true diagonal matrix. An $n \times n$ *unit* or *identity matrix,* denoted normally as $\mathbf{I}$, is a diagonal matrix with $a_{ii} = 1$, $i = 1, \ldots, n$. A *null matrix* or *zero matrix,* designated as $\mathbf{0}$, has $a_{ij} = 0$ for all i and j.

If two matrices $\mathbf{A}$ and $\mathbf{B}$ are of the same dimension, they are defined to be *equal matrices* if $a_{ij} = b_{ij}$ for all i and j. The *sum of two equidimensional matrices* $\mathbf{A}$ and $\mathbf{B}$ is a matrix with the same dimension, defined as

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{2.18}$$

where $c_{ij} = a_{ij} + b_{ij}$ for all i and j. The *difference* between two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension is defined as the matrix

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \tag{2.19}$$

where $c_{ij} = a_{ij} - b_{ij}$ for all i and j. For matrices having the same dimension, the following identities are valid:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = \mathbf{A} + \mathbf{B} + \mathbf{C} \tag{2.20}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{2.21}$$

Multiplication of a matrix by a scalar is defined as

$$\alpha\mathbf{A} = \mathbf{C} \tag{2.22}$$

where $c_{ij} = \alpha a_{ij}$.

Let $\mathbf{A}$ be an $m \times p$ matrix and let $\mathbf{B}$ be a $p \times n$ matrix, written in the form

$$\mathbf{A} \equiv \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix} \qquad \mathbf{B} \equiv [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n]$$

where the $\mathbf{a}_i^T$, $i = 1, \ldots, m$, are row vectors with p elements and the $\mathbf{b}_i$, $i = 1, \ldots, n$, are column vectors with p elements. Then the *matrix product* of $\mathbf{A}$ and $\mathbf{B}$ is defined as the $m \times n$ matrix

$$\mathbf{C} = \mathbf{AB} \tag{2.23}$$

where

$$\mathbf{C} = \begin{bmatrix} \mathbf{a}_1^T\mathbf{b}_1 & \mathbf{a}_1^T\mathbf{b}_2 & \cdots & \mathbf{a}_1^T\mathbf{b}_n \\ \mathbf{a}_2^T\mathbf{b}_1 & \mathbf{a}_2^T\mathbf{b}_2 & \cdots & \mathbf{a}_2^T\mathbf{b}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_m^T\mathbf{b}_1 & \mathbf{a}_m^T\mathbf{b}_2 & \cdots & \mathbf{a}_m^T\mathbf{b}_n \end{bmatrix} \tag{2.24}$$

or $c_{ij} = \mathbf{a}_i^T\mathbf{b}_j$. The *scalar product* $\mathbf{a}^T\mathbf{b}$ for two vectors $\mathbf{a} = [a_1, a_2, \ldots, a_p]^T$ and $\mathbf{b} = [b_1, b_2, \ldots, b_p]^T$ is defined as

$$\mathbf{a}^T\mathbf{b} = a_1b_1 + a_2b_2 + \cdots + a_pb_p \tag{2.25}$$

It is important to note that the product of two matrices is defined only if the number of columns in the first matrix equals the number of rows in the second matrix. It is clear from the definition that, in general,

$$\mathbf{AB} \neq \mathbf{BA} \tag{2.26}$$

In fact, the products **AB** and **BA** are defined only if both **A** and **B** are square and of equal dimension.

The following identities are valid, assuming that the matrices have proper dimensions:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC} \tag{2.27}$$

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC} \tag{2.28}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{2.29}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{2.30}$$

If $a_{ij} = a_{ji}$ for all i and j, the matrix **A** is called *symmetric;* i.e., $\mathbf{A} = \mathbf{A}^T$. If $a_{ij} = -a_{ji}$ for all i and j, the matrix **A** is called *skew-symmetric;* i.e., $\mathbf{A} = -\mathbf{A}^T$. Note that in this case, $a_{ii} = 0$, for all i.

Consider an $m \times p$ matrix **A**. If linear combinations of the rows of the matrix are nonzero; i.e., if

$$\mathbf{A}^T\boldsymbol{\alpha} \neq \boldsymbol{0} \tag{2.31}$$

for all $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T \neq \boldsymbol{0}$, then the rows of **A** are said to be *linearly independent*. Otherwise, if

$$\mathbf{A}^T\boldsymbol{\alpha} = \boldsymbol{0} \tag{2.32}$$

for at least one $\boldsymbol{\alpha} \neq \boldsymbol{0}$, then the rows of **A** are said to be *linearly dependent* and at least one of them can be written as a linear combination of the others.

The *row rank (column rank)* of a matrix **A** is defined as the largest number of linearly independent rows (columns) in the matrix. The row and the column ranks of any matrix are equal. Each of them can thus be called the *rank* of the matrix. A square matrix with linearly independent rows (columns) is said to have full rank. When a square matrix does not have full rank, it is called *singular*. For a nonsingular matrix there is an *inverse,* denoted by $\mathbf{A}^{-1}$, such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{2.33}$$

The following identities are valid:

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{2.34}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{2.35}$$

A special nonsingular matrix that arises often in kinematics is called an *orthogonal matrix,* with the property that[†]

$$\mathbf{A}^{-1} = \mathbf{A}^T \tag{2.36}$$

Therefore, for an orthogonal matrix,

$$\mathbf{A}^T\mathbf{A} = \mathbf{I} \tag{2.37}$$

Since constructing the inverse of a nonsingular matrix is generally time-consuming, it is important to know when a matrix is orthogonal. In this special case, the inverse is trivially constructed by using Eq. 2.36.

2.2.2 Algebraic Vector Operations

The algebraic representation of vectors provides a powerful tool for vector algebra. A reader who is not familiar with this notation and arithmetic may not realize at first its ease of use and flexibility. However, after learning how to operate with algebraic vectors, the reader will find that the traditional geometrical vector operation is rigid and limited for formula manipulation.

An algebraic vector is defined as a column matrix. When an algebraic vector represents a geometric vector in three-dimensional space, the algebraic vector has three components and is called a 3-vector. However, algebraic vectors with more than three components will also be defined and employed in this text.

A 3-vector **a** was shown in Eq. 2.17 in terms of its *xyz* components. The components of a vector can be specified in terms of the other coordinate systems besides the *xyz* coordinate system, such as the $x'y'z'$ or $\xi\eta\zeta$ system. In order not to restrict the following notation to the *xyz* components of a vector, we show the components of vectors **a** and **b** as

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \tag{2.38}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2.39}$$

and thus the vector sum of Eq. 2.1 becomes, in algebraic notation,

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{2.40}$$

It is also true that $\vec{a} = \vec{b}$ if the components of the vectors are equal; i.e., if $\mathbf{a} = \mathbf{b}$. Multiplication of a vector $\vec{a}$ by a scalar α occurs component by component, so the vector $\alpha\vec{a}$ is described by the column vector $\alpha\mathbf{a}$. A null or zero vector, denoted by $\boldsymbol{0}$, has all of its components equal to zero.

The scalar product of two vectors may be expressed in algebraic form as

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a} = a_1b_1 + a_2b_2 + a_3b_3 \tag{2.41}$$

[†]The correct terminology would have been *orthonormal* instead of *orthogonal*.

Note that two vectors **a** and **b** are orthogonal if

$$\mathbf{a}^T\mathbf{b} = \mathbf{0} \tag{2.42}$$

A *skew-symmetric* matrix associated with a vector **a** is defined as

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2.43}$$

Note that the *tilde* placed over a vector indicates that the components of the vector are used to generate a skew-symmetric matrix. Now the vector product $\vec{a} \times \vec{b}$ in Eq. 2.9 can be written in component form as

$$\mathbf{c} = \tilde{\mathbf{a}}\mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{2.44}$$

For later use, it is helpful to develop some standard properties of the tilde operation. First note that

$$\tilde{\mathbf{a}}^T = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \tag{2.45}$$

Also, for a scalar α,

$$\alpha\tilde{\mathbf{a}} = (\widetilde{\alpha\mathbf{a}}) \tag{2.46}$$

For any vectors **a** and **b**, a direct calculation shows that

$$\tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a} \tag{2.47}$$

Direct calculation may also be done to show that

$$\tilde{\mathbf{a}}\mathbf{a} = \mathbf{0} \tag{2.48}$$

Hence, by Eq. 2.45,

$$\mathbf{a}^T\tilde{\mathbf{a}}^T = -\mathbf{a}^T\tilde{\mathbf{a}} = \mathbf{0}^T \tag{2.49}$$

It can also be verified by direct calculation that

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I} \tag{2.50}$$

where **I** is a 3 × 3 identity matrix. Also,

$$(\widetilde{\tilde{\mathbf{a}}\mathbf{b}}) = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T \tag{2.51}$$

$$= \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}} \tag{2.52}$$

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} + \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{b}}\tilde{\mathbf{a}} + \mathbf{b}\mathbf{a}^T \tag{2.53}$$

It can also be verified by direct calculation that

$$(\widetilde{\mathbf{a} + \mathbf{b}}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \tag{2.54}$$

Table 2.1 should assist the reader in becoming familiar with the algebraic notation.

Example 2.1

Test the validity of Eq. 2.47 with two vectors $\mathbf{a} = [-2, 1, -3]^T$ and $\mathbf{b} = [1, -2, 4]^T$.

Solution The product $\tilde{\mathbf{a}}\mathbf{b}$ is computed as follows:

$$\tilde{\mathbf{a}}\mathbf{b} = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$$

The product $\tilde{\mathbf{b}}\mathbf{a}$ is computed similarly:

$$\tilde{\mathbf{b}}\mathbf{a} = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$$

It can be seen that $\tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a}$.

Example 2.1a

For vectors **a** and **b**, verify Eq. 2.51.

Solution The product $\tilde{\mathbf{a}}\mathbf{b}$ was found to be $[-2, 5, 3]^T$. Therefore,

$$\widetilde{\tilde{\mathbf{a}}\mathbf{b}} = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$$

The right-hand side of Eq. 2.51 is computed as follows:

$$\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} [-2, 1, -3] - \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} [1, -2, 4]$$

$$= \begin{bmatrix} -2 & 1 & -3 \\ 4 & -2 & 6 \\ -8 & 4 & -12 \end{bmatrix} - \begin{bmatrix} -2 & 4 & -8 \\ 1 & -2 & 4 \\ -3 & 6 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$$

which verifies the validity of Eq. 2.51.

Example 2.2

Show that

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

Solution Using algebraic vector notation, we write the left-hand side of this expression as

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c} + \tilde{\mathbf{b}}\tilde{\mathbf{c}}\mathbf{a} + \tilde{\mathbf{c}}\tilde{\mathbf{a}}\mathbf{b}$$

Employing Eq. 2.50 for $\tilde{\mathbf{a}}\tilde{\mathbf{b}}$, $\tilde{\mathbf{b}}\tilde{\mathbf{c}}$, and $\tilde{\mathbf{c}}\tilde{\mathbf{a}}$, the above terms become

$$(\mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I})\mathbf{c} + (\mathbf{c}\mathbf{b}^T - \mathbf{b}^T\mathbf{c}\mathbf{I})\mathbf{a} + (\mathbf{a}\mathbf{c}^T - \mathbf{c}^T\mathbf{a}\mathbf{I})\mathbf{b}$$

TABLE 2.1 Vector Terms in Geometric and Algebraic Forms

Geometric	Algebraic
$\vec{a}$	$\mathbf{a}$
$\vec{a} + \vec{b}$	$\mathbf{a} + \mathbf{b}$
$\alpha\vec{a}$	$\alpha\mathbf{a}$
$\vec{a} \cdot \vec{b}$	$\mathbf{a}^T\mathbf{b}$
$\vec{a} \times \vec{b}$	$\tilde{\mathbf{a}}\mathbf{b}$
$\vec{a} \cdot (\vec{b} \times \vec{c})$	$\mathbf{a}^T\tilde{\mathbf{b}}\mathbf{c}$
$(\vec{b} \times \vec{c}) \cdot \vec{a}$	$(\tilde{\mathbf{b}}\mathbf{c})^T\mathbf{a}$ $(= -\mathbf{c}^T\tilde{\mathbf{b}}\mathbf{a})$
$\vec{a} \times (\vec{b} \times \vec{c})$	$\tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c}$
$(\vec{a} \times \vec{b}) \times \vec{c}$	$\widetilde{\tilde{\mathbf{a}}\mathbf{b}}\mathbf{c}$

or

$$\mathbf{b}\mathbf{a}^T\mathbf{c} - \mathbf{a}^T\mathbf{b}\mathbf{c} + \mathbf{c}\mathbf{b}^T\mathbf{a} - \mathbf{b}^T\mathbf{c}\mathbf{a} + \mathbf{a}\mathbf{c}^T\mathbf{b} - \mathbf{c}^T\mathbf{a}\mathbf{b}$$

Observe that $\mathbf{b}\mathbf{a}^T\mathbf{c} = \mathbf{a}^T\mathbf{c}\mathbf{b}$, since $\mathbf{a}^T\mathbf{c}$ is a scalar and can be placed to the left or to the right of vector $\mathbf{b}$. Since $\mathbf{a}^T\mathbf{c} = \mathbf{c}^T\mathbf{a}$, then $\mathbf{b}\mathbf{a}^T\mathbf{c} = \mathbf{c}^T\mathbf{a}\mathbf{b}$. Similarly, it can be shown that $\mathbf{c}\mathbf{b}^T\mathbf{a} = \mathbf{a}^T\mathbf{b}\mathbf{c}$, $\mathbf{a}\mathbf{c}^T\mathbf{b} = \mathbf{b}^T\mathbf{c}\mathbf{a}$, and the identity is proved to be zero.

Consider three vectors $\mathbf{a} = [a_1, a_2, a_3]^T$, $\mathbf{b} = [b_1, b_2, b_3]^T$, and $\mathbf{c} = [c_1, c_2, c_3]^T$. For these vectors, the following representations in matix form will be used in this text:

$$\mathbf{A} = [\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \tag{2.55}$$

$$\mathbf{A}^T = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^T = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{2.56}$$

The algebraic representation of vectors allows one to define vectors with more than three components; i.e., vectors with higher dimension than 3. A vector with n components is called an n-vector. For example, the vector $\mathbf{a} = [a_1, a_2, a_3]^T$ is a 3-vector, and

$$\mathbf{d} = [a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3]^T = [\mathbf{a}^T, \mathbf{b}^T, \mathbf{c}^T]^T \tag{2.57}$$

is a 9-vector. Note that the right side of Eq. 2.57 is a column vector. In this text, for clarification purposes in particular cases, the dimension of a vector is shown as a superscript; e.g., vector $\mathbf{d}$ for Eq. 2.57 can be shown as $\mathbf{d}^{(9)}$.

A matrix can be represented in terms of its subvectors and submatrices. For example, the 3×4 matrix $\mathbf{C}$ can be represented as

$$\mathbf{C} = [\mathbf{a}, \mathbf{A}] \tag{2.58}$$

where $\mathbf{a}$ is a 3-vector and $\mathbf{A}$ is a 3×3 matrix. Vector $\mathbf{a}$ represents the elements of the first column of $\mathbf{C}$, and the matrix $\mathbf{A}$ represents the elements in columns 2, 3, and 4 of $\mathbf{C}$.

Example 2.3

If $\mathbf{C} = [\mathbf{a}, \mathbf{A}]$ and $\mathbf{D} = [\mathbf{b}, \mathbf{B}]$ are two 3×4 matrices, express $\mathbf{C}\mathbf{D}^T$ and $\mathbf{C}^T\mathbf{D}$ in terms of $\mathbf{a}$, $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{B}$.

Solution The product $\mathbf{CD}^T$ yields a 3×3 matrix:

$$\mathbf{CD}^T = [\mathbf{a}, \mathbf{A}]\begin{bmatrix}\mathbf{b}^T \\ \mathbf{B}^T\end{bmatrix} = \mathbf{ab}^T + \mathbf{AB}^T$$

and the product $\mathbf{C}^T\mathbf{D}$ yields a 4×4 matrix:

$$\mathbf{C}^T\mathbf{D} = \begin{bmatrix}\mathbf{a}^T \\ \mathbf{A}^T\end{bmatrix}[\mathbf{b}, \mathbf{B}] = \begin{bmatrix}\mathbf{a}^T\mathbf{b} & \mathbf{a}^T\mathbf{B} \\ \mathbf{A}^T\mathbf{b} & \mathbf{A}^T\mathbf{B}\end{bmatrix}$$

2.3 VECTOR AND MATRIX DIFFERENTIATION

In the kinematics and dynamics of mechanical systems, vectors representing the positions of points on bodies, or equations describing the geometry or the dynamics of the motion, are often functions of time or some other variables. In analyzing these equations, time derivatives or partial derivatives with respect to some variables of the vectors and equations are needed. In this section, these derivatives are defined and the notation used in the text is explained.

2.3.1 Time Derivatives

In analyzing velocities and accelerations, time derivatives of vectors that locate points or bodies or equations that describe the geometry of motion must often be calculated. Consider a vector $\mathbf{a} \equiv \mathbf{a}(t) = [a_1(t), a_2(t), a_3(t)]^T$, where t is a scalar parameter that may play the role of time or some other variable. The *time derivative of a vector* **a** is denoted by

$$\frac{d}{dt}\mathbf{a}(t) = \left[\frac{d}{dt}a_1(t), \frac{d}{dt}a_2(t), \frac{d}{dt}a_3(t)\right]^T \equiv \dot{\mathbf{a}} \tag{2.59}$$

Thus, for vectors that are written in terms of their components in a fixed Cartesian coordinate system, the derivative of a vector is obtained by differentiating its components. The derivative of the sum of two vectors gives

$$\frac{d}{dt}(\mathbf{a}(t) + \mathbf{b}(t)) = \dot{\mathbf{a}} + \dot{\mathbf{b}} \tag{2.60}$$

which is completely analogous to the ordinary differentiation rule that the derivative of a sum is the sum of the derivatives. The following vector forms of the product rule of differentiation can also be verified:

$$\frac{d}{dt}(\alpha\mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}} \tag{2.61}$$

$$\frac{d}{dt}(\mathbf{a}^T\mathbf{b}) = \dot{\mathbf{a}}^T\mathbf{b} + \mathbf{a}^T\dot{\mathbf{b}} \tag{2.62}$$

$$\frac{d}{dt}(\tilde{\mathbf{a}}\mathbf{b}) = \dot{\tilde{\mathbf{a}}}\mathbf{b} + \tilde{\mathbf{a}}\dot{\mathbf{b}} \tag{2.63}$$

where $\alpha(t)$ is a scalar function of time. Note also that

$$\dot{\tilde{\mathbf{a}}} = \tilde{\dot{\mathbf{a}}} \tag{2.64}$$

Many uses may be made of these derivative formulas. For example, if the length of a vector $\mathbf{a}(t)$ is fixed, i.e., if $\mathbf{a}(t)^T\mathbf{a}(t) = \mathbf{c}$, then

$$\dot{\mathbf{a}}^T\mathbf{a} = 0 \tag{2.65}$$

If $\mathbf{a}$ is a position vector that locates a given point, then $\dot{\mathbf{a}}$ is the velocity of that point. Hence Eq. 2.65 indicates that the velocity is orthogonal to the position vector when the position vector has a constant magnitude.

The second time derivative of a vector $\mathbf{a} \equiv \mathbf{a}(t)$ is denoted as

$$\frac{d}{dt}\left(\frac{d}{dt}\mathbf{a}(t)\right) \equiv \frac{d}{dt}(\dot{\mathbf{a}}) \equiv \ddot{\mathbf{a}} \tag{2.66}$$

Thus, for vectors that are written in terms of their components in a fixed Cartesian coordinate system, the second time derivative may be calculated in terms of the second time derivatives of the components of the vector.

Just as in the differentiation of a vector function, the *derivative of a matrix* whose components depend on a variable t may be defined. Consider a matrix $\mathbf{A}(t) = [a_{ij}(t)]$. The derivative of $\mathbf{A}(t)$ is defined as

$$\frac{d}{dt}\mathbf{A}(t) = \left[\frac{d}{dt}a_{ij}(t)\right] \equiv \dot{\mathbf{A}} \tag{2.67}$$

With this definition, it can be verified that

$$\frac{d}{dt}(\mathbf{A}(t) + \mathbf{B}(t)) = \dot{\mathbf{A}} + \dot{\mathbf{B}} \tag{2.68}$$

$$\frac{d}{dt}(\mathbf{A}(t)\mathbf{B}(t)) = \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}} \tag{2.69}$$

$$\frac{d}{dt}(\alpha(t)\mathbf{A}(t)) = \dot{\alpha}\mathbf{A} + \alpha\dot{\mathbf{A}} \tag{2.70}$$

$$\frac{d}{dt}(\mathbf{A}(t)\mathbf{a}(t)) = \dot{\mathbf{A}}\mathbf{a} + \mathbf{A}\dot{\mathbf{a}} \tag{2.71}$$

Example 2.4

If $\mathbf{a}$ is a nonzero time-dependent 3-vector, $\mathbf{A} = [\mathbf{a}, \tilde{\mathbf{a}}]$ is a 3×4 matrix, and $\mathbf{C} = \mathbf{A}\mathbf{A}^T$, what is the condition on $\mathbf{a}$ for which $\dot{\mathbf{C}}$ will be a null matrix?

Solution Matrix $\mathbf{C}$ is found to be

$$\mathbf{C} = [\mathbf{a}, \tilde{\mathbf{a}}]\begin{bmatrix} \mathbf{a}^T \\ -\tilde{\mathbf{a}} \end{bmatrix} = \mathbf{a}\mathbf{a}^T - \tilde{\mathbf{a}}\tilde{\mathbf{a}}$$

The time derivative of $\mathbf{C}$ is

$$\dot{\mathbf{C}} = \dot{\mathbf{a}}\mathbf{a}^T + \mathbf{a}\dot{\mathbf{a}}^T - \dot{\tilde{\mathbf{a}}}\tilde{\mathbf{a}} - \tilde{\mathbf{a}}\dot{\tilde{\mathbf{a}}} = \dot{\mathbf{a}}\mathbf{a}^T + \mathbf{a}\dot{\mathbf{a}}^T - \mathbf{a}\dot{\mathbf{a}}^T + \dot{\mathbf{a}}^T\mathbf{a}\mathbf{I} - \dot{\mathbf{a}}\mathbf{a}^T + \mathbf{a}^T\dot{\mathbf{a}}\mathbf{I}$$

where Eq. 2.50 is employed. Since $\dot{\mathbf{a}}^T\mathbf{a} = \mathbf{a}^T\dot{\mathbf{a}}$, after simplification it is found that $\dot{\mathbf{C}} = 2\dot{\mathbf{a}}^T\mathbf{a}\mathbf{I}$. Therefore, $\dot{\mathbf{C}} = \mathbf{0}$ if $\mathbf{a}$ has a constant magnitude; i.e., $\dot{\mathbf{a}}^T\mathbf{a} = 0$.

2.3.2 Partial Derivatives

In dealing with systems of nonlinear differential and algebraic equations in many variables, it is essential that a *matrix calculus* notation be employed. To introduce the notation used here, let $\mathbf{q}$ be a k-vector of real variables and Φ be a scalar differentiable function of $\mathbf{q}$. Using j as column index, the following notation is defined:

$$\Phi_{\mathbf{q}} \equiv \frac{\partial \Phi}{\partial \mathbf{q}} \equiv \left[\frac{\partial \Phi}{\partial q_j}\right]_{(1\times k)} \tag{2.72}$$

Equation 2.72 indicates that the partial derivative of a scalar function with respect to a variable vector is a row vector.

Example 2.5

Vector $\mathbf{q}$ designates four variables as $\mathbf{q} = [x_1, x_2, x_3, x_4]^T$. Find the partial derivative of a scalar function Φ with respect to $\mathbf{q}$ where $\Phi = -x_1 + 3x_2x_4^2$.

Solution Since $\partial\Phi/\partial x_1 = -1$, $\partial\Phi/\partial x_2 = 3x_4^2$, $\partial\Phi/\partial x_3 = 0$, and $\partial\Phi/\partial x_4 = 6x_2x_4^2$; then, using Eq. 2.72, $\Phi_{\mathbf{q}}$ is written as follows:

$$\Phi_{\mathbf{q}} = [-1 \quad 3x_4^2 \quad 0 \quad 6x_2x_4]$$

If $\mathbf{\Phi}(\mathbf{q}) = [\Phi_1(\mathbf{q}), \Phi_2(\mathbf{q}), \ldots, \Phi_m(\mathbf{q})]^T$ is an m-vector of differentiable functions of $\mathbf{q}$, using i as row index and j as column index, the following notation is defined:

$$\mathbf{\Phi}_{\mathbf{q}} \equiv \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}} \equiv \left[\frac{\partial \Phi_i}{\partial q_j}\right]_{(m\times k)} \tag{2.73}$$

Equation 2.73 indicates that the partial derivative of m functions of a k-vector of variables with respect to that vector is defined as an $m \times k$ matrix.

Example 2.6

Vector $\mathbf{q}$ containing six variables is given as $\mathbf{q} = [x_1, y_1, x_2, y_2, x_3, y_3]^T$. Determine the partial derivative of two functions $\mathbf{\Phi} = [\Phi_1, \Phi_2]^T$ with respect to $\mathbf{q}$ where

$$\Phi_1 = x_1 + 3y_1 - x_2 + 2x_3 - y_3$$
$$\Phi_2 = x_1y_1 + y_2 + 2y_3$$

Solution The partial derivative of $\mathbf{\Phi}$ with respect to $\mathbf{q}$ is a 2×6 matrix:

$$\mathbf{\Phi}_{\mathbf{q}} = \begin{bmatrix} 1 & 3 & -1 & 0 & 2 & -1 \\ y_1 & x_1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

Note that the first and the second rows of this matrix contain the partial derivatives of Φ_1 and Φ_2 with respect to $\mathbf{q}$ respectively.

The partial derivative of the scalar product of two n-vector functions $\mathbf{a}(\mathbf{q})$ and $\mathbf{b}(\mathbf{q})$, by careful manipulation, is found to be a row vector:

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{a}^T\mathbf{b}) = \mathbf{b}^T\mathbf{a}_{\mathbf{q}} + \mathbf{a}^T\mathbf{b}_{\mathbf{q}} \tag{2.74}$$

where the dimension of the resultant row vector is the same as the dimension of vector $\mathbf{q}$.

Example 2.7

Vectors **a** and **b** are functions of a single variable α. Determine the partial derivative of $\mathbf{a}^T\mathbf{b}$ with respect to α if

$$\mathbf{a} = \begin{bmatrix} 2\alpha \\ -1 \\ \alpha \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -3 \\ \alpha \\ -\alpha \end{bmatrix}$$

Solution The derivatives of **a** and **b** with respect to α are

$$\mathbf{a}_\alpha = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{b}_\alpha = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Using Eq. 2.74, it is found that

$$\frac{\partial}{\partial \alpha}(\mathbf{a}^T\mathbf{b}) = [-3, \alpha, -\alpha]\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + [2\alpha, -1, \alpha]\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -2\alpha - 7$$

This result can be obtained directly, in order to verify Eq. 2.74, by determining the scalar product $\mathbf{a}^T\mathbf{b}$:

$$\mathbf{a}^T\mathbf{b} = [2\alpha, -1, \alpha]\begin{bmatrix} -3 \\ \alpha \\ -\alpha \end{bmatrix} = -\alpha^2 - 7\alpha$$

Then the partial derivative of $\mathbf{a}^T\mathbf{b}$ with respect to α is found to be $-2\alpha - 7$.

Example 2.8

Vector **q** contains two variables α and β; i.e., $\mathbf{q} = [\alpha, \beta]^T$. Vectors **a** and **b** are functions of **q**, as follows:

$$\mathbf{a} = \begin{bmatrix} \alpha - \beta \\ \alpha + \beta^2 \\ \beta - 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -\alpha^2 + \beta \\ \alpha + 2 \\ -\alpha - \beta^2 \end{bmatrix}$$

Determine the partial derivative of $\mathbf{a}^T\mathbf{b}$ with respect to **q**.

Solution The derivatives of **a** and **b** with respect to **q** are:

$$\mathbf{a}_\mathbf{q} = \begin{bmatrix} 1 & -1 \\ 1 & 2\beta \\ 0 & 1 \end{bmatrix} \qquad \mathbf{b}_\mathbf{q} = \begin{bmatrix} -2\alpha & 1 \\ 1 & 0 \\ -1 & -2\beta \end{bmatrix}$$

Using Eq. 2.74, it is found that

$$\frac{\partial}{\partial \mathbf{q}}(\mathbf{a}^T\mathbf{b}) = [-\alpha^2 + \beta, \alpha + 2, -\alpha - \beta^2]\begin{bmatrix} 1 & -1 \\ 1 & 2\beta \\ 0 & 1 \end{bmatrix}$$

$$+ [\alpha - \beta, \alpha + \beta^2, \beta - 1]\begin{bmatrix} -2\alpha & 1 \\ 1 & 0 \\ -1 & -2\beta \end{bmatrix}$$

$$= [-\alpha^2 + \beta + \alpha + 2, \alpha^2 - \beta + 2\alpha\beta + 4\beta - \alpha - \beta^2]$$
$$+ [-2\alpha^2 + 2\alpha\beta + \alpha + \beta^2 - \beta + 1, \alpha - \beta - 2\beta^2 + 2\beta]$$
$$= [-3\alpha^2 + 2\alpha\beta + \beta^2 + 2\alpha + 3, \alpha^2 + 2\alpha\beta - 3\beta^2 + 4\beta]$$

The partial derivative of the vector product of two n-vector functions $\mathbf{a}(\mathbf{q})$ and $\mathbf{b}(\mathbf{q})$ is found to be

$$\frac{\partial}{\partial \mathbf{q}}(\tilde{\mathbf{a}}\mathbf{b}) = \tilde{\mathbf{a}}\mathbf{b}_q - \tilde{\mathbf{b}}\mathbf{a}_q \tag{2.75}$$

The resultant matrix of Eq. 2.75 is an $n \times m$ matrix, where m is the dimension of $\mathbf{q}$.

Example 2.7a

Evaluate Eq. 2.75 for vectors **a** and **b**.

Solution The derivatives of **a** and **b** with respect to α are already available; therefore

$$\frac{\partial}{\partial \alpha}(\tilde{\mathbf{a}}\mathbf{b}) = \begin{bmatrix} 0 & -\alpha & -1 \\ \alpha & 0 & -2\alpha \\ 1 & 2\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & \alpha & \alpha \\ -\alpha & 0 & 3 \\ -\alpha & -3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2\alpha + 1 \\ -3 + 4\alpha \\ 4\alpha \end{bmatrix}$$

Example 2.9

Vectors **a**, **b**, and **c** are functions of vector **q**. Find $\mathbf{d}_q$ where $\mathbf{d} = \tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c}$.

Solution In expressions such as $\tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c}$ where several functions appear in a nonlinear form, it is helpful to find equivalent forms of the expression. Each equivalent expression should have a different vector appearing at its extreme right. For instance, **d** can be presented in three forms:

$$\mathbf{d} = \tilde{\mathbf{a}}\tilde{\mathbf{b}}\mathbf{c} = -\tilde{\mathbf{a}}\tilde{\mathbf{c}}\mathbf{b} = -(\widetilde{\tilde{\mathbf{b}}\mathbf{c}})\mathbf{a}$$

where Eq. 2.47 has been employed. Now, these identities easily yield the partial derivative of **d** with respect to **q**:

$$\mathbf{d}_q = (\tilde{\mathbf{a}}\tilde{\mathbf{b}})\mathbf{c}_q - (\tilde{\mathbf{a}}\tilde{\mathbf{c}})\mathbf{b}_q - (\widetilde{\tilde{\mathbf{b}}\mathbf{c}})\mathbf{a}_q$$

This approach can be used to verify Eq. 2.75.

PROBLEMS

2.1 Let $\vec{a} = \vec{u}_{(x)} + 2\vec{u}_{(y)} - \vec{u}_{(z)}$ and $\vec{b} = 2\vec{u}_{(x)} - \vec{u}_{(y)} + \vec{u}_{(z)}$. Use the algebraic vector approach to calculate the following:

(a) $\vec{a} + \vec{b}$

(b) $\vec{a} \cdot \vec{b}$

(c) $\alpha\vec{a}$

(d) $\vec{a} \times \vec{b}$

(e) $(\vec{a} - \vec{b}) \times \vec{a}$

2.2 If $\tilde{a}$ and $\vec{b}$ are arbitrary 3-vectors, verify the following identities by direct calculation:

(a) Eq. 2.50

(b) Eq. 2.51

(c) Eq. 2.52

(d) Eq. 2.53

2.3 If $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are 3-vectors, use the algebraic vector approach to show that the following identities are valid:

(a) $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

(b) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

2.4 Show that if $\mathbf{A}$ is a square matrix, the matrices $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ and $\mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ are symmetric and skew-symmetric, respectively.

2.5 Show that any square matrix $\mathbf{A}$ can be uniquely expressed as $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{B}$ and $\mathbf{C}$ are symmetric and skew-symmetric, respectively.

2.6 Show that for an arbitrary angle ϕ, the matrix $\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ is orthogonal.

2.7 Show that for arbitrary angles, ϕ, ψ, θ, and σ, the following matrices are orthogonal c ≡ cos and s ≡ sin):

(a) $\mathbf{A} = \begin{bmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} c\psi & -s\psi\, c\theta & s\psi\, s\theta \\ s\psi & c\psi\, c\theta & -c\psi\, s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} c\psi\, c\sigma - s\psi\, c\theta\, s\sigma & -c\psi\, s\sigma - s\psi\, c\theta\, c\sigma & s\psi\, s\theta \\ s\psi\, c\sigma + c\psi\, c\theta\, s\sigma & -s\psi\, s\sigma + c\psi\, c\theta\, c\sigma & -c\psi\, s\theta \\ s\theta\, s\sigma & s\theta\, c\sigma & c\theta \end{bmatrix}$

2.8 If $\mathbf{e}$ is a 3-vector and e_0 is a scalar, show that

$$\mathbf{A} = (2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})$$

is a 3×3 orthogonal matrix, knowing that $e_0^2 + \mathbf{e}^T\mathbf{e} = 1$.

2.9 Vector $\mathbf{a}$ is a 3-vector and $\mathbf{B}$ is a 3×4 matrix defined as $\mathbf{B} = [\mathbf{a}, \tilde{\mathbf{a}}]$. What is the condition for $\mathbf{A} = \mathbf{B}\mathbf{B}^T$ to be an orthogonal matrix?

2.10 In Prob. 2.9, show that under no condition can matrix $\mathbf{C} = \mathbf{B}^T\mathbf{B}$ be orthogonal.

2.11 Let $\mathbf{B} = [\mathbf{a}, \tilde{\mathbf{a}}]$ and $\mathbf{C} = [\mathbf{a}, -\tilde{\mathbf{a}}]$. Show that $\dot{\mathbf{B}}\mathbf{C}^T = \mathbf{B}\dot{\mathbf{C}}^T$.

2.12 Vector $\mathbf{\Phi}$ contains two functions as follows:

$$\mathbf{\Phi} = \begin{bmatrix} 2x - 3xy + y^2 - xz + yz^2 - 4xyz \\ -x^2 + xy^2 - 2y + 5yz - xz^2 \end{bmatrix}$$

If vector $\mathbf{q}$ is defined as $\mathbf{q} = [x, y, z]^T$, find:

(a) $\mathbf{\Phi}_\mathbf{q}$

(b) $\dot{\mathbf{\Phi}}$

Show that $\dot{\mathbf{\Phi}} = \mathbf{\Phi}_\mathbf{q}\dot{\mathbf{q}}$.

2.13 Use vectors **a**, **b**, and **q** from Example 2.8 and evaluate Eq. 2.75.

2.14 For two 3-vectors **s** and $\boldsymbol{\omega}$, vector $\dot{\mathbf{s}}$ is defined as $\dot{\mathbf{s}} = \tilde{\boldsymbol{\omega}}\mathbf{s}$. Show that

$$\ddot{\mathbf{s}} = -\tilde{\mathbf{s}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{s}$$

2.15 For two 3-vectors **s** and $\boldsymbol{\omega}$ show that

$$\tilde{\mathbf{s}}\tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}}\tilde{\mathbf{s}}\tilde{\mathbf{s}}\boldsymbol{\omega}$$

2.16 Vectors **a** and **b** are defined as $\mathbf{a} = \mathbf{A}_1\mathbf{c}_1$ and $\mathbf{b} = \mathbf{A}_2\mathbf{c}_2$, where

$$\mathbf{A}_i = \begin{bmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{bmatrix} \qquad i = 1, 2 \qquad \mathbf{c}_1 = \begin{bmatrix} 1.2 \\ -0.5 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} -0.3 \\ 0.8 \end{bmatrix}$$

(a) Let $\Phi = \mathbf{a}^T\mathbf{b}$ and $\mathbf{q} = [x_1, y_1, \phi_1, x_2, y_2, \phi_2]^T$. Evaluate $\Phi_\mathbf{q}$ for $\phi_1 = 30°$ and $\phi_2 = 45°$.

(b) Let $\mathbf{d} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$ and $\mathbf{\Phi} = \tilde{\mathbf{a}}\mathbf{d}$. Evaluate $\mathbf{\Phi}_\mathbf{q}$ for $\phi_1 = 30°$, $\phi_2 = 45°$, $x_1 = 6.2$, $y_1 = 1.0$, $x_2 = -1.9$, and $y_2 = 2.3$.

2.17 Let **a** and **b** be two 3-vectors, $\mathbf{B} = [\mathbf{b}, \tilde{\mathbf{b}}]$, and $\mathbf{C} = \mathbf{B}^T\mathbf{a}$. Find $\mathbf{C}_\mathbf{a}$ and $\mathbf{C}_\mathbf{b}$.

2.18 Let **x** be an n-vector of real variables and **A** be a real $n \times n$ matrix. Show that

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \mathbf{x}^T\mathbf{A}^T + \mathbf{x}^T\mathbf{A} = \mathbf{x}^T(\mathbf{A}^T + \mathbf{A})$$

2.19 If the matrix **A** in Prob. 2.18 is symmetric, show that

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^T\mathbf{A}\mathbf{x}) = 2\mathbf{x}^T\mathbf{A}$$