

Lecture Outline 8: Constrained Optimization II

This lecture note is based on Chapter 19 of *Mathematics for Economists* by Simon and Blume.

1. The Meaning of the Multiplier

- One equality constraint: $\mu^*(a) = \frac{d}{da}f(x^*(a), y^*(a))$ measures the rate of change of the optimal value of f with respect to the parameter a .
- The above applies to several equality constraints and inequality constraints.
- The Lagrange multiplier: shadow price

Exercise a) Find the maximum and minimum distance from the origin to the ellipse $x^2 + xy + y^2 = 3$. b) Use Theorem 19.1 and your answer to Exercise 18.2 to estimate the answers to part a.

2. Envelope Theorems

- Unconstrained problem $\frac{d}{da}f(x^*(a); a) = \frac{\partial}{\partial a}f(x^*(a); a)$ (Q: can you prove it?)
- Constrained problem $\frac{d}{da}f(x^*(a); a) = \frac{\partial}{\partial a}L(x^*(a), u(a); a)$

Exercise Use Exercise 18.2 and the Envelope Theorem to estimate the maximum and minimum distance from the origin to the ellipse $x^2 + xy + 0.9y^2 = 3$.

3. Second Order Conditions

- Constrained maximization problems
 - * Let's recall bordered matrix ...
 - * form the bordered matrix with the Hessian of L and $Dh(x^*)$.
 - * If the above bordered matrix is negative definite, then we have a local max.
- Minimization problems: the bordered matrix being positive definite is the sufficient condition for a local min.
- Inequality constraints: only use binding inequality constraints
- Alternative approaches to the bordered Hessian condition: please read it by yourself

Exercise Redo the proof of Theorem 19.7 for the case $\frac{\partial h}{\partial y}(x^*, y^*) = 0$.

4. Smooth Dependence of the Parameters

- Nondegenerate critical point: nonsingular Hessian (Q: what's an example of a degenerate Hessian? $f(x) = -x^4, f'(0) = 0$.)
- Nonsingular Hessian \Rightarrow NDCQ

5. Constraint Qualifications

- More discussion on NDCQ violation: see Chiang & Wainwright 13.2.
- Fritz John Theorem

Exercise Verify that constraint qualification b in Theorem 19.12 is not satisfied in Example 19.9