Lecture Outline 7: Constrained Optimization I: First Order Conditions

This lecture note is based on Chapter 18 of *Mathematics for Economists* by Simon and Blume.

1. Equality Constraints
   - Two variables and one equality constraint: let's consider a utility maximization problem with a budget $\max U(x_1, x_2)$ s.t. $p_1x_1+ p_2x_2 = I$
     * What conditions must hold where the level curve of f is tangent to the constraint set?
     * Lagrangian function and Lagrange multiplier: reducing a constrained optimization problem to an unconstrained one.
     * constraint qualification: (x_1^*, x_2^*) can not be a critical point of the constraint function.
     * Example: $\max f(x, y) = x$ s.t. $h(x, y) = x^3 + y^2 = 0$
     * Example: $\max f(x, y) = x^3 + y^3$ s.t. $g(x, y) = x - y = 0$
   - m equality constraints
     * nondegenerate constraint qualification (NDCQ): Jacobian derivative $Dh(x^*)$ has rank m. (Q: Can you show that this implies that $m \leq n$?)
   - Second-order conditions: Definiteness of bordered matrices (details later)
   - A "Cookbook" procedure using the theorem of Lagrange
     * Set up the Lagrangean function
     * Find the set of all critical points
     * Evaluate the function at each critical point in the set
     * In practice the above procedure USUALLY yield the solutions
   - When and Why would the Lagreanean Method fail?
     * If an optimum exisits but the constraint qualification is not met at the optimum
     * An optimum may not exist

   Exercise 1 *Find the maximum and minimum distance from the origin to the ellipse $x^2+xy+y^2 = 3$. (Hint: Use $x^2 + y^2$ as your objective function.)*

2. Inequality Constraints
   - The sign of the lagrange multiplier matters!
   - Solution can be interior
   - Complementary Slackness Condition: $\lambda \cdot [g(x, y) - b] = 0$
   - NDCQ only involves binding constraints
   - Assume all constraints are binding, what do you get? Relax each constraint in turn, what do you get?

   Exercise 2 *Find the maximizer of $f(x, y) = 2y^2 - x$, subject to the constraints $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.*

3. Mixed Constraints: please read it by yourself
4. Constrained Minimization Problems
   - In stead of $g(x) \leq b$, formulate $g(x) \geq b$
   - Alternative 1: minimizing f = maximizing $-f$
   - Alternative 2: negative Lagrangian multiplier

   Exercise 3 *Check that the NDCQ are satisfied in Example 18.11.*

Exercise 4 *Present a geometric proof that in the problem of minimizing $f(x, y)$ on the constraint set $g(x, y) \geq b$, the gradient of f and the gradient of g point in the same direction at a minimizer for which the constraint is binding.*

5. Kuhn-Tucker Formulation

   - Separate nonnegativity constraint from the rest of the inequality constraints

   Exercise 5 *Write out the Kuhn-Tucker conditions for problem 18.10*

6. Examples and Applications: Please read it by yourself. You can skip the Averch-Johnson Effect.