Lecture Outline 7: Constrained Optimization I: First Order Conditions

This lecture note is based on Chapter 18 of *Mathematics for Economists* by Simon and Blume.

1. Equality Constraints

   - Two variables and one equality constraint: let’s consider a utility maximization problem with a budget \( \max U(x_1, x_2) \) s.t. \( p_1 x_1 + p_2 x_2 = I \)
     * What conditions must hold where the level curve of \( f \) is tangent to the constraint set?
     * Lagrangian function and Lagrange multiplier: reducing a constrained optimization problem to an unconstrained one.
     * Constraint qualification: \( (x_1^*, x_2^*) \) can not be a critical point of the constraint function.
     * Example: \( \max f(x, y) = x \) s.t. \( h(x, y) = x^3 + y^2 = 0 \)
     * Example: \( \max f(x, y) = x^3 + y^3 \) s.t. \( g(x, y) = x - y = 0 \)

   - \( m \) equality constraints
     * Nondegenerate constraint qualification (NDCQ): Jacobian derivative \( Dh(x^*) \) has rank \( m \).
       (Q: Can you show that this implies that \( m \leq n \)?)
     * Second-order conditions: Definiteness of bordered matrices (details later)
     * A "Cookbook" procedure using the theorem of Lagrange
       * Set up the Lagrangean function
       * Find the set of all critical points
       * Evaluate the function at each critical point in the set
       * In practice the above procedure USUALLY yield the solutions

   - When and Why would the Lagrangean Method fail?
     * If an optimum exists but the constraint qualification is not met at the optimum
     * An optimum may not exist

   **Exercise 1** Find the maximum and minimum distance from the origin to the ellipse \( x^2 + xy + y^2 = 3 \). (Hint: Use \( x^2 + y^2 \) as your objective function.)

2. Inequality Constraints

   - The sign of the lagrange multiplier matters!
   - Solution can be interior
   - Complementary Slackness Condition: \( \lambda \cdot [g(x, y) - b] = 0 \)
   - NDCQ only involves binding constraints
   - Assume all constraints are binding, what do you get? Relax each constraint in turn, what do you get?

   **Exercise 2** Find the maximizer of \( f(x, y) = 2y^2 - x \), subject to the constraints \( x^2 + y^2 \leq 1 \), \( x \geq 0 \), \( y \geq 0 \).

3. Mixed Constraints: please read it by yourself

4. Constrained Minimization Problems

   - In stead of \( g(x) \leq b \), formulate \( g(x) \geq b \)
   - Alternative 1: minimizing \( f = \) maximizing \(-f\)
   - Alternative 2: negative Lagrangian multiplier

   **Exercise 3** Check that the NDCQ are satisfied in Example 18.11.
Exercise 4 Present a geometric proof that in the problem of minimizing $f(x, y)$ on the constraint set $g(x, y) \geq b$, the gradient of $f$ and the gradient of $g$ point in the same direction at a minimizer for which the constraint is binding.

5. Kuhn-Tucker Formulation

- Separate nonnegativity constraint from the rest of the inequality constraints

Exercise 5 Write out the Kuhn-Tucker conditions for problem 18.10

6. Examples and Applications: Please read it by yourself. You can skip the Averch-Johnson Effect.