

Lecture Outline 6: Optimization Problem; Unconstrained Optimization

This lecture note is based on Chapter 17 of *Mathematics for Economists* by Simon and Blume.

1. Optimization Problem in \mathbb{R}^n : $\max f(x)$ subject to $x \in D$

- Objective function and constraint set
- Suprema, Infima, Maxima, Minima (Q: What is the difference between Suprema (Infima) and Maxima (Minima)?)
- In a given optimization problem, a solution may fail to exist
- Even if a solution does exist, it need not necessarily be unique
- Every maximization problem may be presented as a minimization problem, and vice versa.
- A transformation of the optimization problem under which the solution remains unaffected (Q: why do you need the transformation to be "strictly increasing" instead of "nondecreasing"?)
- Optimization problem in parametric form
- Some examples: utility maximization, expenditure minimization

Exercise 1 1) Let $D = \mathbb{R}_+$ and $f(x) = x$ for $x \in D$. Solve for maxima. 2) Let $D = [-1, 1]$ and $f(x) = x^2$ for $x \in D$. Solve for maxima.

2. The Objectives of Optimization Theory

- To identify a set of conditions on f and D under which the *existence* of solutions to optimization problems is guaranteed
- To obtain a *characterization* of the set of optimal points
 - * Necessary conditions
 - * Sufficient conditions
 - * Conditions that guarantee uniqueness of solutions

3. Existence of Solutions

- The Weierstrass Theorem: Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \rightarrow \mathbb{R}$ be a continuous function on D . Then f attains a maximum and a minimum on D .
- The Weierstrass Theorem only provides sufficient conditions for the existence of optima

Exercise 2 Let $D = [0, 1]$. Let $f : D \rightarrow \mathbb{R}$ be a strictly increasing function on D , and let $g : D \rightarrow \mathbb{R}$ be a strictly decreasing function on D (That is, if $x, y \in D$ with $x > y$ then $f(x) > f(y)$ and $g(x) < g(y)$). Then f attains a minimum and a maximum on D (at 0 and 1, respectively), as does g (at 1 and 0, respectively). Does $f + g$ necessarily attain a maximum and minimum on D ?

4. Unconstrained Optimization

- Definition: local max(min) vs. global max(min)
- First order conditions: $\frac{\partial F}{\partial x_i}(x^*) = 0$ for all i
 - * critical points: points at which $\frac{\partial F}{\partial x_i}(x) = 0$ for all i or $\frac{\partial F}{\partial x_i}(x)$ is not defined. Not necessarily interior points.
- Second order conditions: check the definiteness of the Hessian $D^2F(x^*)$
- Check the boundary points
- Note the differences between the sufficient conditions and necessary conditions for the second order conditions (definiteness vs. semidefiniteness).

Exercise 3 For function $f(x, y) = x^4 + x^2 - 6xy + 3y^2$, find the critical points and classify these as local max, local min, saddle point, or "can't tell".

5. Global Maxima and Minima

- Relate global max(min) to concaveness(convexness)
 - * convex open set: see P506 for definition
 - * Semidefiniteness (for all points in a whole ball around x^*) is enough for sufficient conditions!

Exercise 4 In the above exercise, is there global max or global min?