Lecture Outline 6: Quadratic Forms, Definite Matrices, and Unconstrained Optimization

This lecture note is based on Chapter 16 and 17 of *Mathematics for Economists* by Simon and Blume.

1. Quadratic Forms $\sum_{i \le j} a_{ij}x_i x_j$ (in matrix form, $x^T Ax$)
2. Definiteness of Quadratic Forms
   - Definition:
     - positive definite: > 0 at points other than the origin
     - negative definite: < 0 at points other than the origin
     - indefinite: ≥ 0 or ≤ 0 at points other than the origin
     - positive semidefinite: ≥ 0 at points other than the origin
     - negative semidefinite: ≤ 0 at points other than the origin
   - Geometric representation
   - Definite symmetric matrices
   - Principal minors of a matrix
     - k^{th} order principal submatrix
     - principal minor: the determinant of k^{th} order principal submatrix
     - leading principal submatrix and leading principal minor
     - how can we tell the definiteness of a matrix? Check the leading principal minors!
     - checking for the semidefinitenss of a matrix takes more work... we need to check every principal minor.
   - Definitenss of diagonal matrices

Exercise *Determine the definiteness of the following symmetric matrices: a)* $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$*;*

b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

3. Linear Constraints and Bordered Matrices
   - a 2×2 matrix with linear constraint $Ax_1 + Bx_2 = 0$ is pos. (neg.) definite iff the bordered matrix $\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$ has a neg.(pos.) determinant.
   - The result holds for the general problem: to check the definiteness of the matrix $x^T Ax$ with linear constraint $Bx = 0$, we check the bordered matrix $\begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}$'s last $n - m$ leading princial minors.
   - Example: one constraint

Exercise *Determine the definiteness of the following constrained quadratics:*

a. $Q(x_1,x_2) = x_1^2 + 2x_1x_2 - x_2^2$, *subject to* $x_1 + x_2 = 0$.
b. $Q(x_1,x_2,x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, *subject to* $x_1 + x_2 + x_3 = 0$ *and* $x_1 + x_2 - x_3 = 0$.

4. Unconstrained Optimization

- Definition: local max(min) vs. global max(min)
- First order conditions
  - * critical point and interior point
  - * $\frac{\partial F}{\partial x_i}(x^*) = 0$ for all i
- Second order conditions: check the definitenss of of the Hessian $D^2F(x^*)$
- Note the differences between the sufficient conditions and necessary conditions for the second order conditions (definitenss vs. semidefiniteness).

Exercise *For function* $x^4 + x^2 - 6xy + 3y^2$, *find the critical points and classify these as local max, local min, saddle point, or "can't tell".*

5. Global Maxima and Minima

- Relate global max(min) to concaveness(convexness)
  - * convex open set: see P506 for definition
  - * Semidefiniteness (for all points in a whole ball around x^*) is enough for sufficient conditions!

Exercise *In the above exercise, is there global max or global min?*

6. Economic Applications: Please read it by yourself.