Lecture Outline 6: Optimization Problem; Unconstrained Optimization

This lecture note is based on Chapter 17 of *Mathematics for Economists* by Simon and Blume.

1. Optimization Problem in $\mathbb{R}^n$: $\max f(x)$ subject to $x \in D$
   - Objective function and constraint set
   - Suprema, Infima, Maxima, Minima (Q: What is the difference between Suprema (Infima) and Maxima (Minima)?)
   - In a given optimization problem, a solution may fail to exist
   - Even if a solution does exist, it need not necessarily be unique
   - Every maximization problem may be presented as a minimization problem, and vice versa.
   - A transformation of the optimization problem under which the solution remains unaffected (Q: why do you need the transformation to be "strictly increasing" instead of "nondecreasing"?)
   - Optimization problem in parametric form
   - Some examples: utility maximization, expenditure minimization

**Exercise 1**
1) Let $D = \mathbb{R}_+$ and $f(x) = x$ for $x \in D$. Solve for maxima. 2) Let $D = [-1, 1]$ and $f(x) = x^2$ for $x \in D$. Solve for maxima.

2. The Objectives of Optimization Theory
   - To identify a set of conditions on $f$ and $D$ under which the existence of solutions to optimization problems is guaranteed
   - To obtain a characterization of the set of optimal points
     * Necessary conditions
     * Sufficient conditions
     * Conditions that guarantee uniqueness of solutions

3. Existence of Solutions
   - The Weierstrass Theorem: Let $D \subset \mathbb{R}^n$ be compact, and let $f : D \to \mathbb{R}$ be a continuous function on $D$. Then $f$ attains a maximum and a minimum on $D$.
   - The Weierstrass Theorem only provides sufficient conditions for the existence of optima

**Exercise 2** Let $D = [0, 1]$. Let $f : D \to \mathbb{R}$ be a strictly increasing function on $D$, and let $g : D \to \mathbb{R}$ be a strictly decreasing function on $D$ (That is, if $x, y \in D$ with $x > y$ then $f(x) > f(y)$ and $g(x) < g(y)$). Then $f$ attains a minimum and a maximum on $D$ (at 0 and 1, respectively), as does $g$ (at 1 and 0, respectively). Does $f + g$ necessarily attain a maximum and minimum on $D$?

4. Unconstrained Optimization
   - Definition: local max(min) vs. global max(min)
   - First order conditions: $\frac{\partial F}{\partial x_i}(x^*) = 0$ for all $i$
     * critical points: points at which $\frac{\partial F}{\partial x_i}(x) = 0$ for all $i$ or $\frac{\partial F}{\partial x_i}(x)$ is not defined. Not necessarily interior points.
   - Second order conditions: check the definiteness of the Hessian $D^2 F(x^*)$
   - Check the boundary points
   - Note the differences between the sufficient conditions and necessary conditions for the second order conditions (definiteness vs. semidefiniteness).
Exercise 3  For function \( f(x, y) = x^4 + x^2 - 6xy + 3y^2 \), find the critical points and classify these as local max, local min, saddle point, or "can't tell".

5. Global Maxima and Minima

- Relate global max(min) to concaveness(convexness)
  * convex open set: see P506 for definition
  * Semidefiniteness (for all points in a whole ball around \( x^* \)) is enough for sufficient conditions!

Exercise 4  In the above exercise, is there global max or global min?