

Lecture Outline 5: Implicit Functions and Quadratic Forms

This lecture note is based on Chapter 15 and 16 of *Mathematics for Economists* by Simon and Blume.

1. Implicit Functions of Their Derivatives

- Implicit function $G(x, y) = c$
- Does $y(x)$ exist close to (x_0, y_0) ? Is $y(x)$ differentiable?
- Implicit function theorem $y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}$ (apply the Chain Rule)
- The theorem applies to implicit functions with several exogenous variables

Exercise 1 Prove that the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x in a neighborhood of $(x, y) = (5, 2)$. Then, estimate the y value which corresponds to $x = 4.8$.

2. System of Implicit Functions

- Linear systems (see example 15.14): m endogenous variables, n exogenous variables
- Nonlinear systems
 - * Linearize by taking the derivative, apply the linear theorem to this linearized system, and transfer these results back to the original nonlinear system.
 - * Implicit Function Theorem in the most generous form

Exercise 2 One solution of the system $x^3y - z = 1$, $x + y^2 + z^3 = 6$ is $x = 1$, $y = 2$, $z = 1$. Use calculus to estimate the corresponding x and y when $z = 1.1$.

Exercise 3 Consider the system of equations $y^2 + 2u^2 + v^2 - xy = 15$, $2y^2 + u^2 + v^2 + xy = 38$, at the solutions $x = 1$, $y = 4$, $u = 1$, $v = -1$. Think of u and v as exogenous and x and y as endogenous. Use calculus to estimate the values of x and y that correspond to $u = 0.9$ and $v = -1.1$.

3. Application — Comparative Statics Using the Implicit Function Theorem: I leave this to your own reading. I think it's a good idea for us to see an economic application of the implicit function theorem. However, I find that this section is somewhat difficult to follow. Let me give you some big-picture comments to help you understand what is going on.

- The Model
 - (a) For those who saw it in intermediate micro, this is exactly the Edgeworth Box framework. You have two consumers (each with their own utility function) and two goods. To begin with, consumer 1 has all of good 1 and consumer 2 has all of good 2. Consumer 1 can exchange some good 1 with consumer 2 for some good 2, thus making both consumers better off.
 - (b) We have five endogenous variables: four of them tell us how much each consumer gets of each good (x_1, x_2, y_1, y_2), and one (p) that tells us how much good 1 can be obtained (by consumer 2 from consumer 1) for a unit of good 2.
 - (c) We have three exogenous variables (also known as parameters):
 - * the parameter (α) telling us the relative weight of $u(x)$ and $u(y)$ in each consumer's utility function.
 - * the total endowment (e_1) of good 1.
 - * the total endowment (e_2) of good 2.
 - (d) We have five equations in our five unknowns. Two equations (41) and (42) come from consumer 1's maximization problem and his budget constraint. Two more equations (43) and (44) come from consumer 2's maximization problem and her budget constraint. The last two equations (45) and (46) come from the total consumption of each good being equal to the total endowment of each good in the economy. That's six equations, but one of them is redundant: we can ignore (44) because it is implied by three of the other equations. That means we have five equations determining five endogenous variables.

- The Analysis
 - (a) Now that we understand the model, the whole point of the application is to show how the Implicit Function Theorem can be useful to help us do comparative statics. In particular, we ask "Suppose the total endowment e_2 of good 2 increases. How does this affect each consumer's consumption bundle and the price of the two goods?" In other words, if we change the exogenous variable e_2 but hold α and e_1 constant, by how much do x_1 , x_2 , y_1 , y_2 , and p change?
 - (b) The answers, derived from the Implicit Function Theorem, are given in (53). We can see that the third equation and the fifth equation show us derivatives that are unambiguously positive, no matter what the shapes of the consumer utility functions $u_1(x)$ and $u_2(x)$: the price of good 1 and the quantity of good 2 consumed by consumer 1 both go up when e_2 goes up. The other three expressions have ambiguous signs that depend on the amounts of concavity of the two consumer utility functions.
 - (c) Note that the quantities r_1 , r_2 , R_1 , R_2 , and D are defined for notational convenience, so that the expressions in (53) are simpler to write. This is a standard technique. Note in particular that we have defined them in ways that guarantee that each of them are positive, which makes it easier to examine the expressions in (53) to see if they are positive or negative. These five quantities all depend on the first and second derivatives of $u_1(x)$ and $u_2(x)$.
 - (d) Quadratic Forms $\sum_{i \leq j} a_{ij} x_i x_j$ (in matrix form, $x^T A x$)

4. Definiteness of Quadratic Forms

- Definition:
 - * positive definite: > 0 at points other than the origin
 - * negative definite: < 0 at points other than the origin
 - * indefinite: ≥ 0 or ≤ 0 at points other than the origin
 - * positive semidefinite: ≥ 0 at points other than the origin
 - * negative semidefinite: ≤ 0 at points other than the origin
- Geometric representation
- Definite symmetric matrices
- Principal minors of a matrix
 - * k^{th} order principal submatrix
 - * principal minor: the determinant of k^{th} order principal submatrix
 - * leading principal submatrix and leading principal minor
 - * how can we tell the definiteness of a matrix? Check the leading principal minors!
 - * checking for the semidefiniteness of a matrix takes more work... we need to check every principal minor.
- Definiteness of diagonal matrices

Exercise 4 Determine the definiteness of the following symmetric matrices: a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$

5. Linear Constraints and Bordered Matrices

- a 2×2 matrix with linear constraint $Ax_1 + Bx_2 = 0$ is pos. (neg.) definite iff the bordered matrix $\begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$ has a neg.(pos.) determinant.

- The result holds for the general problem: to check the definiteness of the matrix $x^T Ax$ with linear constraint $Bx = 0$, we check the bordered matrix $\begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}$'s last $n - m$ leading principal minors.
- Example: one constraint

Exercise 5 Determine the definiteness of the following constrained quadratics:

(a) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.

(b) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.