Lecture Outline 4: Functions and Calculus of Several Variables

This lecture note is based on Chapter 13 and 14 of *Mathematics for Economists* by Simon and Blume.

1. Functions Between Euclidean Spaces: I leave this to your own reading.

2. Geometric Representation of Functions
   - Two variables: the slice method
     **Exercise 1** Use the slice method to sketch the graphs of the following functions: a) \( z = -x^2 - y^2 \); b) \( z = y - x^2 \); c) \( z = ye^{-x} \).
   - Level curves
     **Exercise 2** If you had the graph of \( z = f(x, y) \), how would you use it to draw level curves of \( f \)?
     **Exercise 3** Describe another example where a map with level curves occurs naturally in real life. What are the implications of curves being close together in this situation?
   - Planar level sets in economics
     * Isoquants
     * Indifference curves
     * Level sets

3. Special Kinds of Functions
   - Linear functions on \( \mathbb{R}^k \): \( f(x) = Ax \) for all \( x \in \mathbb{R}^k \)
   - Quadratic forms: \( Q(x_1, \ldots, x_k) = \sum_{i,j=1}^{k} a_{ij}x_i x_j \), or in matrix form, \( x'Ax \) (Q: what is the geometric representation of a general quadratic form on \( \mathbb{R}^2 \)?)
   - Monomials and polynomials
     **Exercise 4** Write the following functions in matrix form:
     (a) \( f(x_1, x_2, x_3) = 2x_1 - 3x_2 + 5x_3 \)
     (b) \( f(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 \)

4. Continuous Functions: I leave this to your own reading.

5. Vocabulary of Functions
   - Basic vocabulary: domain, target space, range, image, preimage
   - Onto functions and one-to-one functions: surjective vs. injective
   - Inverse functions
   - Compositions of functions
     **Exercise 5** For each of the following functions, what is the domain and the range of \( f \)? Which ones are one-to-one? For those which are one-to-one, write the expression for the inverse. Which ones are onto?
     (a) \( f(x) = 3x - 7 \)
     (b) \( f(x) = x^3 - x \)

6. Calculus
   - Total derivatives
     * Geometric interpretation
7. Directional Derivatives and Gradients of Functions from \( R^n \) to \( R^1 \)

- Derivative of \( F \) at \( x^* \) in the direction of \( v \): \( DF_{x^*} \cdot v \) (\( DF_{x^*}: \) row vector of partial derivatives; \( v \): direction)
- Gradient: column vector of partial derivatives
- Normalize the length of \( v \) to be 1
- Gradient points into the the direction in which \( F \) increases most rapidly.

**Exercise 9** Consider the function \( y^2 e^{3x} \). In what direction should one move from the point \((0,3)\) to increase the value of this function most rapidly? Express your answer as a vector of length 1.

8. Explicit Functions from \( R^n \) to \( R^m \)

- Jacobian derivatives \( DF_{x^*}: m \times n \) matrix of partial derivatives
- The Chain Rule of functions from \( R^1 \) to \( R^m \): \( g'(t) = DF(a(t)) \cdot a'(t) \): \( g'(t) \) is a \( m \times 1 \) vector; \( a'(t) \) is a \( n \times 1 \) vector
- The Chain Rule of functions from \( R^n \) to \( R^m \): \( Dg(t) = DF(a(t)) \cdot Da(t) \): \( Dg(t) \) is a \( m \times s \) vector; \( Da(t) \) is a \( n \times s \) vector

**Exercise 10** Given that \( G(x, y) = (x^2 + y^2) \) and \( F(u, v) = (u + v, u^2) \), compute the Jacobian derivative matrix of \( F(G(x, y)) \) at the point \((x, y) = (1, 1)\).

9. Higher Order Derivatives

- Continuously Differentiable Functions and Cross partial Derivatives
- Hessian Matrix \( D^2 f \)
- Young’s Theorem: the order of differentiation does NOT matter.

**Exercise 11** Compute the Hessian matrix for each of the following functions. Verify that each is a symmetric matrix.
(a) \( 4x^2 y - 3xy^3 + 6x \)
(b) \( e^{2x+3y} \)