Lecture Outline 3: Proofs, Limits, and Sets

This lecture note is based on Appendix 1 and Chapter 12 of Mathematics for Economists by Simon and Blume.

1. Sets and Numbers: I leave this to your own reading.
2. Proofs
   - Direct (Deductive) Proof
   - Indirect Proof or Proof by Contradiction
     * Converse: \( A \implies B \)'s converse is \( B \implies A \)
     * If and only if: \( A \iff B \)
     * Contrapositive: see rule 1 of lecture outline 1.
     * Proof by contradiction: to prove to prove \( A \implies B \) is to prove \( \neg(B) \implies \neg(A) \)
   - Inductive Proof
     * Prove statement \( P(1) \) is true.
     * Prove \( P(n) \implies P(n + 1) \).
     * Only applies to propositions about or indexed by integers.

Exercise  Show that \( \sqrt{3} \) is an irrational number.

3. Sequence of Real Numbers (Sequence in \( R^1 \))
   - A sequence \( \{x_1, x_2, \ldots, x_n, \ldots\} \)
   - Limit of a sequence \( \lim_{n \to \infty} x_n = r \)
   - Compare: accumulation point. (Q: If a sequence has multiple accumulation points, does there exist a limit?)
   - A sequence can have at most one limit. (Q: prove it by contradiction?)
   - Algebraic Properties of Limits
     * Limits of sequences are preserved by algebraic operations.
     * Check out the proofs of the theorems: What types of proofs are these? Can you prove them yourselves?

Exercise  Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a sequence of real numbers that converges to \( x_0 \) and that all \( x_n \) and \( x_0 \) are nonzero.

   a. Prove that there is a positive number \( B \) such that \( |x_n| \geq B \) for all \( n \).
   b. Using a, prove that \( \{ \frac{1}{x_n} \} \) converges to \( \{ \frac{1}{x_0} \} \).

4. Sequences in \( R^m \)
   - A sequence of vector in \( R^m \) converges if and only if all \( m \) sequences of its components converge in \( R^1 \).
   - Results about sequences in \( R^1 \) apply to sequences in \( R^1 \).

5. Open Sets
   - Open: no boundary
   - Is \( \{x \in R^2 : 0 < x_1 < 1, x_2 = 0\} \) an open set?
   - Any union of open sets is open.
   - The finite intersection of open sets is open. (Q: why not infinite?)
6. Closed Sets
   - Closed: must contain all its boundary points.
   - Closedness and openness are complementary.
   - Any intersection of closed sets is closed.
   - The finite union of closed sets is closed.

   **Exercise** Show that closed intervals in $\mathbb{R}^1$ — sets of the form \( \{x : a \leq x \leq b\} \) for fixed numbers $a$ and $b$ — are closed sets.

   **Exercise** Is \( \{(x, y) : -1 < x < 1, y = 0\} \) open or closed? Explain.

7. Compact Sets
   - Closed and bounded (Q: Can you give an example of a closed but unbounded set?)
   - Bolzano-Weierstrass theorem: Any sequence defined on a compact set must contain a subsequence that actually converges.

   **Exercise** Prove that every finite set is compact.

   **Exercise** Brainstorm examples of open, closed, and compact sets.